

**1. Polynomials A**  
**2. Polynomials B**  
**3. Polynomials C**

These three classes will cover the theory of polynomials. Polynomials A will include factorization, Vieta's formulas, and elementary symmetric polynomials. Polynomials B will include the Identity Theorem and the Lagrange Interpolation Formula. Polynomials C will include irreducibility and polynomial functional equations.

*Example Problem:* Factor the polynomial  $a^3(b - c) + b^3(c - a) + c^3(a - b)$ .

*Example Problem:* If  $P(x)$  denotes a polynomial of degree  $n$  such that  $P(k) = k/(k + 1)$  for  $k = 0, 1, 2, \dots, n$ , determine  $P(n + 1)$ . (USAMO, 1975)

*Example Problem:* Let  $a_1, a_2, \dots, a_n$  be distinct integers. Prove that the polynomial

$$(x - a_1)(x - a_2) \cdots (x - a_n) - 1$$

is irreducible over the integers.

**4. Group Theory**

This class will introduce group theory (as well as abstract mathematics) with applications. The material in this class will also help establish results in number theory.

**5. Similar Triangles/Power of a Point**

This class will highlight the use of similar triangles and Power of a Point, two simple but powerful techniques in Euclidean geometry.

*Example Problem:* Let  $ABCD$  be a cyclic quadrilateral with  $AB = 6$ ,  $BC = 9$ , and  $CD = 8$ . Diagonals  $AC$  and  $BD$  meet at  $M$ , such that  $BM = 2DM$ . Find  $AD$ .

*Example Problem:* Given circles  $\omega_1$  and  $\omega_2$  intersecting at points  $X$  and  $Y$ , let  $\ell_1$  be a line through the center of  $\omega_1$  intersecting  $\omega_2$  at points  $P$  and  $Q$ , and let  $\ell_2$  be a line through the center of  $\omega_2$  intersecting  $\omega_1$  at points  $R$  and  $S$ . Prove that if  $P, Q, R$ , and  $S$  lie on a circle, then the center of this circle lies on line  $XY$ . (USAMO, 2009)

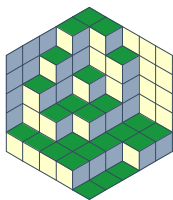
**6. Geometric Transformations**

This class will introduce the use and applications of geometric transformations, such as homothety and spiral similarity.

*Example Problem:* Circles  $\Gamma_1$  and  $\Gamma_2$  are internally tangent at  $P$ , with  $\Gamma_1$  as the larger circle. A line intersects  $\Gamma_1$  at  $A$  and  $D$ , and  $\Gamma_2$  at  $B$  and  $C$ , so that points  $A, B, C$ , and  $D$  lie on the line in that order. Show that  $\angle APB = \angle CPD$ .

*Example Problem:* On the sides of an arbitrary triangle  $ABC$ , triangles  $ABR, BCP, CAQ$  are constructed externally with  $\angle CBP = \angle CAQ = 45^\circ$ ,  $\angle BCP = \angle ACQ = 30^\circ$ ,  $\angle ABR = \angle BAR = 15^\circ$ . Prove that  $\angle QRP = 90^\circ$  and  $QR = RP$ . (IMO, 1975)





### 7. Geometry of the Triangle

This class will present fundamental results on the geometry of triangle, including the major centers of the triangle. There will be an emphasis on trigonometric relations.

*Example Problem:* A line bisects both the perimeter and area of a triangle. Prove that the line passes through the incenter of the triangle.

*Example Problem:* Given triangle  $ABC$ , show that the circumcenter lies on the incircle if and only if  $\cos A + \cos B + \cos C = \sqrt{2}$ .

### 8. Concurrency & Collinearity

This class will present problems involving concurrent lines and collinear points, and the tools to solve them, such as Ceva's theorem and Menelaus's theorem.

*Example Problem:* Let  $ABCD$  and  $AB'C'D'$  be two squares, oriented the same way. Prove that  $BB'$ ,  $CC'$ , and  $DD'$  are concurrent.

*Example Problem:* Let the incircle of triangle  $ABC$  touch sides  $BC$ ,  $CA$ , and  $AB$  at  $D$ ,  $E$ , and  $F$  respectively. Let  $\Gamma$ ,  $\Gamma_1$ ,  $\Gamma_2$ , and  $\Gamma_3$  denote the circumcircles of triangles  $ABC$ ,  $AEF$ ,  $BDF$ , and  $CDE$ , respectively. Let  $\Gamma$  and  $\Gamma_1$  intersect at  $A$  and  $P$ ,  $\Gamma$  and  $\Gamma_2$  intersect at  $B$  and  $Q$ , and  $\Gamma$  and  $\Gamma_3$  intersect at  $C$  and  $R$ . Prove that  $PD$ ,  $QE$ , and  $CR$  are concurrent. (Canada, 2007)

### 9. Modular Arithmetic A

### 10. Modular Arithmetic B

### 12. Modular Arithmetic C

These three classes will cover the theory of modular arithmetic. Modular Arithmetic A will focus on AIME level number theory problems. Modular Arithmetic B will introduce results such as Fermat's Little theorem and Euler's theorem. Modular Arithmetic C will cover quadratic residues, order, and primitive roots.

*Example Problem:* How many integers  $N$  less than 1000 can be written as the sum of  $j$  consecutive positive odd integers from exactly 5 values of  $j \geq 1$ ? (AIME, 2006)

*Example Problem:* The number  $85^9 - 21^9 + 6^9$  is divisible by an integer between 2000 and 3000. Compute that integer. (ARML, 1991)

*Example Problem:* Let  $a$ ,  $b$ , and  $c$  be positive integers that are pairwise relatively prime, and that satisfy  $a^2 - ab + b^2 = c^2$ . Show that every prime factor of  $c$  is of the form  $6k + 1$ , where  $k$  is a positive integer.

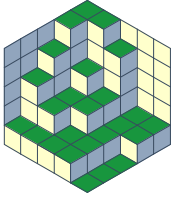
### 11. Diophantine Equations

A *Diophantine equation* is an equation where we seek integer solutions. We will use the techniques presented in the modular arithmetic classes to solve such equations.

*Example Problem:* Determine all non-negative integral solutions  $(n_1, n_2, \dots, n_{14})$  if any, apart from permutations, of the Diophantine equation

$$n_1^4 + n_2^4 + \dots + n_{14}^4 = 1599.$$





# Art of Problem Solving

## WOOT 2009–10

### Class Descriptions

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(USAMO, 1979)

#### 13. Combinatorics A

This class will introduce basic counting principles, such as distributions and the principle of Inclusion-Exclusion.

*Example Problem:* How many solutions does the equation  $x_1 + x_2 + x_3 + x_4 + x_5 = 12$  have in positive integers? In non-negative integers?

*Example Problem:* A permutation  $\{x_1, x_2, \dots, x_{2n}\}$  of the set  $\{1, 2, \dots, 2n\}$  where  $n$  is a positive integer, is said to have property  $T$  if  $|x_i - x_{i+1}| = n$  for at least one  $i$  in  $\{1, 2, \dots, 2n - 1\}$ . Show that, for each  $n$ , there are more permutations with property  $T$  than without. (IMO, 1989)

#### 14. Coloring Arguments

This class will demonstrate the use of coloring arguments to solve problems in combinatorics.

*Example Problem:* Two opposite squares of an  $8 \times 8$  chessboard are removed. Can the remaining chessboard be tiled by dominoes?

*Example Problem:* An  $8 \times 8$  chessboard is tiled by 21  $1 \times 3$  rectangles and a single  $1 \times 1$  square. What are the possible locations of the  $1 \times 1$  square?

#### 15. Graph Theory

This class will cover problems in graph theory, a topic which has appeared frequently on mathematical olympiads.

*Example Problem:* At a certain mathematical conference, every pair of mathematicians are either friends or strangers. At mealtime, every participant eats in one of two large dining rooms. Each mathematician insists upon eating in a room which contains an even number of his or her friends. Prove that the number of ways that the mathematicians may be split between the two rooms is a power of two. (USAMO, 2008)

#### 16. Combinatorics B

This class will cover olympiad-level problems based on concepts introduced in the three previous combinatorics classes.



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