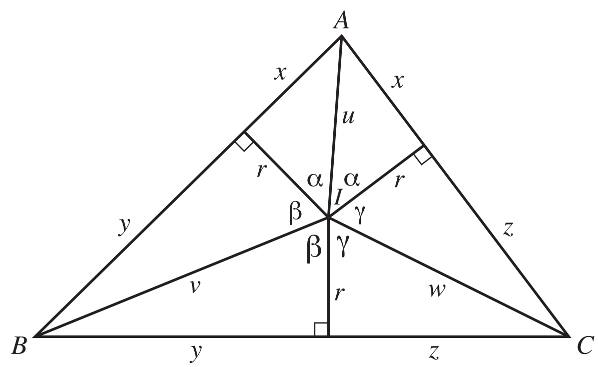
A Proof of Heron's Formula



Let I be the center of the incircle of $\triangle ABC$. Let $a = y + z$, $b = x + z$, and $c = x + y$ be the lengths of the sides opposite A, B, and C, respectively, and let $s = x + y + z$ be the semiperimeter of the triangle. Clearly $2\alpha + 2\beta + 2\gamma = 2\pi$, so $\alpha + \beta + \gamma = \pi$. Now notice that

$$(r + ix)(r + iy)(r + iz) = (ue^{i\alpha})(ve^{i\beta})(we^{i\gamma}) = uvwe^{i(\alpha+\beta+\gamma)} = uvwe^{i\pi} = -uvw.$$

Therefore

$$0 = \text{Im}[(r + ix)(r + iy)(r + iz)] = r^2(x + y + z) - xyz,$$

so

$$r = \sqrt{\frac{xyz}{x + y + z}} = \sqrt{\frac{(s - a)(s - b)(s - c)}{s}}.$$

Thus the area of $\triangle ABC$ is

$$\frac{ra}{2} + \frac{rb}{2} + \frac{rc}{2} = rs = \sqrt{s(s - a)(s - b)(s - c)},$$

which is Heron's formula.

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