On the Expected Winding Number of a Random Walk on the Unit Lattice

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1 Introduction

Given a graph G and a vertex v in G, consider the following process: select a random neighbor of v and move to it. Select and move to a random neighbor of the new vertex, and so on. We call the sequence of vertices visited during this process a *random walk* on G. The behavior of random walks on finite graphs is relatively well understood. The classical results on the steady state properties of these walks are summarized by Lovász [8].

Random walks have applications in graph theory and are involved in a number of algorithms in computer science. For instance, they can be used to solve 2-satisfiability problems in $O(n^2)$ time [9]. They are also involved in the study of Brownian motion [11] and the behavior of polymers [7].

Several new developments in random walks on infinite graphs have been due to techniques involving generating functions as exemplified by Bousquet-Mélou and Schaeffer [3]. They characterized the behavior of walks on the slit plane, the unit lattice without the negative x-axis. These methods may be applicable to more general problems on random walks.

Some recent studies have focused on the *winding number* of a random walk. Given a random walk σ starting at $(1,1)$ on the graph G with vertices defined by $\mathbb{Z}^2$ and edges connecting lattice points separated by distance 1, the *winding number* ω of σ is the number of signed complete rotations the walk has made about $(1/2,1/2)$. A walk from $(1,1)$ to $(3,3)$ with winding number 1 is shown in Figure 1.

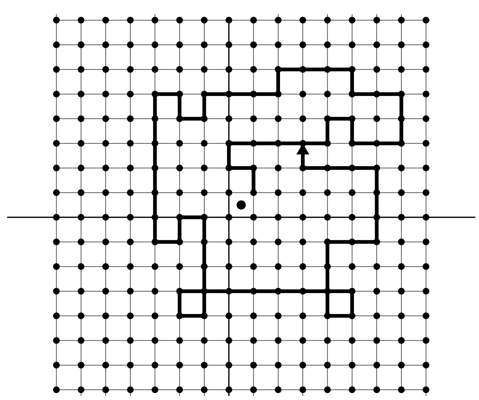

Figure 1: A walk which winds once around $(1/2,1/2)$.

The winding number of the continuous random walk was studied by Berger [2], who showed that the expected root-mean-square winding angle $\sqrt{E\left[\theta^2 \mid n\right]}$ of a continuous random walk is asymptotic to $\frac{\log n}{2}$. This result can be applied to modeling the entanglement of a polymer about an obstacle, as investigated by Drossel and Kardar [5]. However, no exact results are yet known regarding this problem.

Despite the known results on the continuous winding number, the discrete version appears to be unstudied. In this paper, we solve this interesting case. As we show in Appendix A, the expected number of steps it takes to wind once around the origin is infinite. Therefore, we investigate the RMS expected value of the winding number as a function of the number of steps. We first bound the expectation and then derive an exact expression for its explicit formula. In our method, we divide the expectation into three components and find recursive formulas for each component by considering its expectation at any step in terms of its value at a previous step.

2 Classical Results

In this section, we present some useful classical results on infinite random walks.

Definition. A random walk is said to be *recurrent* if it returns to its starting point with probability 1. Otherwise, it is said to be *transient*.

One major result is Pólya's Theorem on the recurrence or transience of random walks on $\mathbb{Z}^n$. A proof is given in [4].

Theorem 1 (Pólya). *The random walk on $\mathbb{Z}^n$ is recurrent if $n \leq 2$ and transient if $n \geq 3$.*

Theorem 2. *The expected squared distance from the origin after n steps of a random walk on $\mathbb{Z}^k$ is $E\left[r^2 \mid n\right] = n$.*

Proof. This result can be proven by considering the generating function for these walks

$$f(x_1, \ldots, x_k) = \frac{\left(\sum_{i=1}^{k}(x_i + 1/x_i)\right)^n}{(2k)^n}$$

and extracting the expectation of x_i^2 given by $\frac{\partial}{\partial x_i}\left(x_i \frac{\partial}{\partial x_i} \left(f(x_1, \ldots, x_k) \right) \big|_{(x_1,\ldots,x_k)=(1,\ldots,1)} \right)$. $\square$

3 Bounds on the Expected Winding Number

In this section, we give bounds on the *root mean square* (RMS) expectation of ω. By this we mean the square root of the expectation of the square of the number of winds.

Let us first reframe the problem. Perform a spiral similarity on $\mathbb{Z}^2$ with a 45° clockwise rotation and ratio $\sqrt{2}$ about $(1/2, 1/2)$. Relabeling $(1/2, 1/2)$ as the origin and considering a new Cartesian coordinate system centered there, we see that there is a bijection between standard random walks on $\mathbb{Z}^2$ starting at $(1, 1)$ and random walks with steps in $\{(1, 1), (1, -1), (-1, 1), (-1, -1)\}$ starting at $(1, 0)$. Thus, the corresponding rotations about $(1/2, 1/2)$ and $(0, 0)$ are equivalent, so we now enumerate rotations around $(0, 0)$ by walks with diagonal steps starting at $(1, 0)$.

3.1 Divergence of the Expectation

Theorem 3. *For any constant k, there exists a sufficiently large n such that $E\left[w^2 \mid n\right] > k$.*

Proof. Denote by $P[\omega]$ the probability that a random walk on the diagonal lattice beginning at $(1, 0)$ has made w full (signed) rotations about $(0, 0)$ upon its first return.

By Pólya's Theorem, we know that any random walk on the diagonal lattice returns to its starting point infinitely many times with probability 1. Conditioning on the event that the walk has just returned to $(1, 0)$ for the l^{th} time, with a winding number of k on the previous visit, we use the fact that $P[\omega] = P[-\omega]$ by symmetry to find

$$E\Big[\omega^2 \mid l \text{ returns} \wedge k \text{ previous winds}\Big]$$
$$= \sum_{\omega=-\infty}^{\infty} P[\omega](k+\omega)^2 = P[0]\, k^2 + \sum_{\omega=1}^{\infty} P[\omega]\left((k+\omega)^2 + (k-\omega)^2\right)$$
$$= P[0]\, k^2 + \sum_{\omega=1}^{\infty} 2\, P[\omega]\left(\omega^2 + k^2\right) = \sum_{\omega=-\infty}^{\infty} P[\omega]\left(\omega^2 + k^2\right) = k^2 + \sum_{\omega=-\infty}^{\infty} P[\omega]\omega^2.$$

The expected change to ω between visits to $(1, 0)$ is independent of k, so, by the linearity of expectation, we obtain

$$E\left[\omega^2 \mid l\right] = E\left[\omega^2 \mid l-1\right] + \sum_{\omega=-\infty}^{\infty} P[\omega]\, \omega^2 = l \sum_{\omega=-\infty}^{\infty} P[\omega]\, \omega^2,$$

meaning that $E\left[\omega^2 \mid l\right]$ exceeds any finite bound for a sufficiently large l, since $\sum_{\omega=-\infty}^{\infty} P[\omega]\, \omega^2$ is a positive constant. Therefore, we have a lower bound of $\omega(1)$ on E. □

3.2 Upper RMS Bound of $O\left(n^{1/4}\right)$

Theorem 4. *The RMS expectation of ω after n steps is $\sqrt{E\left[\omega^2 \mid n\right]} = O(n^{1/4})$.*

Proof. Let us define a new variable c by the equation

$$\begin{aligned} c = {} & \text{Number of steps from the positive } x\text{-axis to the upper half-plane} \\ & + \text{Number of steps from the lower half-plane to the positive } x\text{-axis} \\ & - \text{Number of steps from the positive } x\text{-axis to the lower half-plane} \\ & - \text{Number of steps from the upper half-plane to the positive } x\text{-axis}. \end{aligned}$$

This counter c counts all steps of the form shown in Figure 2. Steps in the counterclockwise direction make a positive contribution, and steps in the clockwise direction make a negative one. The quantities of these steps uniquely define the winding number according to

$$\omega = \begin{cases} \left\lfloor \frac{c}{2} \right\rfloor & c \geq 0 \\ \left\lceil \frac{c}{2} \right\rceil & c < 0 \end{cases},$$

since each full rotation changes c by 2 in the appropriate direction, and no other paths change the magnitude of c by more than 1. The values of ω and c are thus asymptotically equivalent up to a multiplicative constant; we study c because it is more convenient.

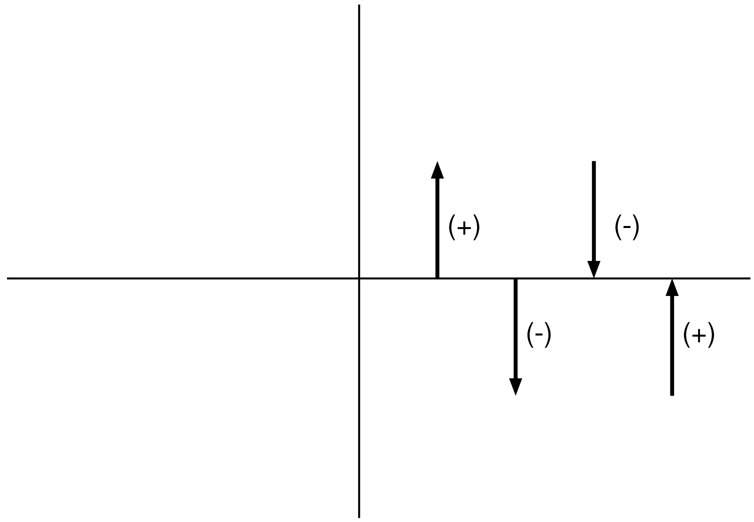

Figure 2: The counter c counts steps of the form shown.

Consider a path which begins at $(a, 0)$ and ends on the x-axis. Let q_a^+ (respectively q_a^-) denote the probability that this path increases c by 1 (respectively decreases c by 1). By symmetry, $q_a^+ = q_a^-$, so we simply write q_a.

Fix k as the value of c and $(a, 0)$ as the position after $l - 1$ returns to the x-axis. Note

$$\begin{aligned} E\left[c^2 \mid l \wedge \text{ previous } c = k\right] &= p_a\, k^2 + q_a(k+1)^2 + q_a(k-1)^2 \\ &= (p_a + 2q_a)k^2 + 2q_a = k^2 + 1 - p_a \le k^2 + 1. \end{aligned}$$

Thus, we see by the linearity of expectation that

$$E\left[c^2 \mid l\right] \le \sum_{k=-\infty}^{\infty} (k^2+1)P[k] = E\left[c^2 \mid l-1\right] + 1 \text{ and thus by induction } E\left[c^2 \mid l\right] \le l. \quad (1)$$

Let r_l be the probability that the random walk returns to the x-axis l times after n steps, and let k_i be the probability that the walk is on the origin after $2i$ steps. Then, we have

$$E\left[l \mid n\right] = \sum_{i=0}^{\infty} r_i\, i = \sum_{i=0}^{\lfloor n/2 \rfloor} k_i = \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{\binom{2i}{i}}{2^{2i}},$$

since $k_i = \frac{\binom{2i}{i}}{2^{2i}}$. Thus, using our result in (1) we obtain

$$E\left[c^2\right] = \sum_{i=0}^{\infty} r_i \, E\left[c^2 \mid i\right] \le \sum_{i=0}^{\infty} r_i \, i = \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{\binom{2i}{i}}{4^i}. \tag{2}$$

Now, applying Stirling's approximation to find $\frac{\binom{2i}{i}}{4^i} \approx \frac{1}{\sqrt{\pi i}}$ and substituting into (2), we have

$$\sqrt{E\left[w^2 \mid n\right]} = O\Big(\sqrt{E\left[c^2 \mid n\right]}\Big) \le O\left(\sqrt{\sum_{i=0}^{\lfloor n/2 \rfloor} \frac{\binom{2i}{i}}{4^i}}\right) \approx O\left(\sqrt{\sum_{i=0}^{\lfloor n/2 \rfloor} \frac{1}{\sqrt{\pi i}}}\right) = O\left(n^{1/4}\right). \quad \square$$

4 Exact Results

Here too we decide to study c instead of ω for convenience. Before giving an explicit formula, we introduce some new notation. Let $c = r + t$, where r and t are given by

$$\begin{aligned} r = {} & \text{Number of steps from the positive } x\text{-axis to the upper half-plane} \\ & - \text{Number of steps from the positive } x\text{-axis to the lower half-plane} \end{aligned}$$

and

$$\begin{aligned} t = {} & \text{Number of steps from the lower half-plane to the positive } x\text{-axis} \\ & - \text{Number of steps from the upper half-plane to the positive } x\text{-axis.} \end{aligned}$$

Definition. Define the points $X_1 = (0, 1)$, $X_2 = (0, -1)$ and the half-lines $L_1 = \{(x, y) : x > 0, y = 1\}$, $L_2 = \{(x, y) : x > 0, y = -1\}$, and $L_0 = \{(x, y) : x > 0, y = 0\}$. These sets of points are useful in characterizing the changes in r and t.

Definition. Denote the position of the walk after i steps by the random variable $p^{(i)}$. Denote the values of r, t, and c after i steps by the random variables $r^{(i)}$, $t^{(i)}$, and $c^{(i)}$, respectively. Define $\mathcal{X}_1^{(i)}$ as the event that $p^{(i)} \in X_1$, and define $\mathcal{X}_2^{(i)}$, $\mathcal{L}_0^{(i)}$, $\mathcal{L}_1^{(i)}$, and $\mathcal{L}_2^{(i)}$ analogously.

To calculate $E\left[\left(c^{(n)}\right)^2\right] = E\left[\left(r^{(n)}\right)^2\right] + 2\,E\left[r^{(n)}\,t^{(n)}\right] + E\left[\left(t^{(n)}\right)^2\right]$, we first find each term.

4.1 Expectation of r^2

Theorem 5. *The expected value* $E\left[\left(r^{(x)}\right)^2\right]$ *of* r^2 *after* x *steps is given by*

$$E\left[\left(r^{(x)}\right)^2\right] = 1 + \sum_{i=1}^{\left\lfloor \frac{x-1}{2} \right\rfloor} \frac{\binom{2i}{i}}{2^{2i}} \left(\frac{1}{2} + \frac{\binom{2i}{i}}{2^{2i+1}} \right).$$

Proof. First note that $E\left[\left(r^{(2n+2)}\right)^2\right] = E\left[\left(r^{(2n+1)}\right)^2\right]$, since r can only change immediately after a walk reaches the positive x-axis, which occurs after an even number of steps. Therefore, we only need consider the case where $x = 2n+1$.

Fix k as the value of r after $2\,n$ steps. Note that

$$\begin{aligned} E\left[\left(r^{(2n+1)}\right)^2 \mid r^{(2n)} = k\right] &= \left(1 - P\left[\mathcal{L}_0^{(2n)} \wedge r^{(2n)} = k\right]\right) k^2 + P\left[\mathcal{L}_0^{(2n)} \wedge r^{(2n)} = k\right] (k^2+1) \\ &= k^2 + P\left[\mathcal{L}_0^{(2n)} \wedge r^{(2n)} = k\right]. \end{aligned}$$

Summing over all possible k, we find that

$$\begin{aligned} E\left[\left(r^{(2n+1)}\right)^2\right] &= \sum_{k=-\infty}^{\infty} P\left[r^{(2n)} = k\right] E\left[\left(r^{(2n+1)}\right)^2 \mid r^{(2n)} = k\right] \\ &= \sum_{k=-\infty}^{\infty} P\left[r^{(2n)} = k\right] \left(k^2 + P\left[\mathcal{L}_0^{(2n)} \wedge r^{(2n)} = k\right]\right) \\ &= E\left[\left(r^{(2n)}\right)^2\right] + P\left[\mathcal{L}_0^{(2n)}\right] = E\left[\left(r^{(2n-1)}\right)^2\right] + \frac{\binom{2n}{n}}{2^{2n}} \left(\frac{1}{2} + \frac{\binom{2n}{n}}{2^{2n+1}}\right) \\ &= 1 + \sum_{i=1}^{n} \frac{\binom{2i}{i}}{2^{2i}} \left(\frac{1}{2} + \frac{\binom{2i}{i}}{2^{2i+1}}\right), \end{aligned}$$

where the last equality results from a simple induction and the fact that $E\left[r^{(1)}\right] = 1$. □

4.2 Expectation of t^2

Theorem 6. *The expected square value* $E\left[\left(t^{(x)}\right)^2\right]$ *of* t *after* $x > 1$ *steps is given by*

$$E\left[\left(t^{(x)}\right)^2\right] = \sum_{i=0}^{\lfloor \frac{x}{2} \rfloor - 1} \left(\frac{1}{2} P\left[\mathcal{X}_1^{(2i+1)}\right] + P\left[\mathcal{L}_1^{(2i+1)}\right]\right)$$
$$- \sum_{i=0}^{\lfloor \frac{x}{2} \rfloor - 1} \left(2P\left[\mathcal{L}_1^{(2i+1)}\right] E\left[t^{(2i+1)} \mid \mathcal{L}_1^{(2i+1)}\right] + P\left[\mathcal{X}_1^{(2i+1)}\right] E\left[t^{(2i+1)} \mid \mathcal{X}_1^{(2i+1)}\right]\right).$$

Proof. Note that t^2 can only change after an odd number of steps. Therefore, $E\left[\left(t^{(2n+1)}\right)^2\right] = E\left[\left(t^{(2n)}\right)^2\right]$. We only consider the case where $x = 2n+2$, since the other case follows.

Fix k as the value of t after $2n+1$ steps. Then, because the value of t can only change after a step leaving X_1, X_2, L_1, or L_2, we have that

$$\begin{aligned}
E\left[\left(t^{(2n+2)}\right)^2 \mid t^{(2n+1)} = k\right] =& \left(1 - P\left[\mathcal{X}_1^{(2n+1)}\right] - P\left[\mathcal{X}_2^{(2n+1)}\right] - P\left[\mathcal{L}_1^{(2n+1)}\right] - P\left[\mathcal{L}_2^{(2n+1)}\right]\right) k^2 \\
&+ P\left[\mathcal{X}_1^{(2n+1)}\right] \left(\frac{1}{4}(k-1)^2 + \frac{3}{4}k^2\right) + P\left[\mathcal{X}_2^{(2n+1)}\right] \left(\frac{1}{4}(k+1)^2 + \frac{3}{4}k^2\right) \\
&+ P\left[\mathcal{L}_1^{(2n+1)}\right] \frac{1}{2}\left((k-1)^2 + k^2\right) + P\left[\mathcal{L}_2^{(2n+1)}\right] \frac{1}{2}\left((k+1)^2 + k^2\right) \\
=& k^2 + \frac{1}{4}P\left[\mathcal{X}_1^{(2n+1)}\right] + \frac{1}{4}P\left[\mathcal{X}_2^{(2n+1)}\right] + \frac{1}{2}P\left[\mathcal{L}_1^{(2n+1)}\right] + \frac{1}{2}P\left[\mathcal{L}_2^{(2n+1)}\right] \\
&- \frac{1}{2}P\left[\mathcal{X}_1^{(2n+1)}\right] k + \frac{1}{2}P\left[\mathcal{X}_2^{(2n+1)}\right] k - P\left[\mathcal{L}_1^{(2n+1)}\right] k + P\left[\mathcal{L}_2^{(2n+1)}\right] k,
\end{aligned}$$

where the above probabilities are conditioned on the event $t^{(2n+1)} = k$. Summing over k and applying Bayes' Theorem, we have

$$\begin{aligned}
E\left[\left(t^{(2n+2)}\right)^2\right] =& \sum_{k=-\infty}^{\infty} P\left[t^{(2n+1)} = k\right] E\left[\left(t^{(2n+2)}\right)^2 \mid t^{(2n+1)} = k\right] \\
=& E\left[\left(t^{(2n+1)}\right)^2\right] + \frac{1}{4}P\left[\mathcal{X}_1^{(2n+1)}\right] + \frac{1}{4}P\left[\mathcal{X}_2^{(2n+1)}\right] + \frac{1}{2}P\left[\mathcal{L}_1^{(2n+1)}\right] + \frac{1}{2}P\left[\mathcal{L}_2^{(2n+1)}\right] \\
&- P\left[\mathcal{L}_1^{(2n+1)}\right] E\left[t^{(2n+1)} \mid \mathcal{L}_1^{(2n+1)}\right] - \frac{1}{2} P\left[\mathcal{X}_1^{(2n+1)}\right] E\left[t^{(2n+1)} \mid \mathcal{X}_1^{(2n+1)}\right] \\
&+ P\left[\mathcal{L}_2^{(2n+1)}\right] E\left[t^{(2n+1)} \mid \mathcal{L}_2^{(2n+1)}\right] + \frac{1}{2} P\left[\mathcal{X}_2^{(2n+1)}\right] E\left[t^{(2n+1)} \mid \mathcal{X}_2^{(2n+1)}\right],
\end{aligned}$$

But we know that $E\left[t^{(2n+1)} \mid \mathcal{L}_2^{(2n+1)}\right] = -E\left[t^{(2n+1)} \mid \mathcal{L}_1^{(2n+1)}\right]$ and $E\left[t^{(2n+1)} \mid \mathcal{X}_2^{(2n+1)}\right] = -E\left[t^{(2n+1)} \mid \mathcal{X}_1^{(2n+1)}\right]$, since the paths satisfying these conditions are reflections of each other about the x-axis. Also applying $E\left[\left(t^{(2n+1)}\right)^2\right] = E\left[\left(t^{(2n)}\right)^2\right]$, we have that

$$\begin{aligned}
E\left[\left(t^{(2n+2)}\right)^2\right] =& E\left[\left(t^{(2n)}\right)^2\right] + \frac{1}{2}P\left[\mathcal{X}_1^{(2n+1)}\right] + P\left[\mathcal{L}_1^{(2n+1)}\right] \\
&- 2P\left[\mathcal{L}_1^{(2n+1)}\right] E\left[t^{(2n+1)} \mid \mathcal{L}_1^{(2n+1)}\right] - P\left[\mathcal{X}_1^{(2n+1)}\right] E\left[t^{(2n+1)} \mid \mathcal{X}_1^{(2n+1)}\right] \\
=& \sum_{i=0}^{n} \left(\frac{1}{2}P\left[\mathcal{X}_1^{(2i+1)}\right] + P\left[\mathcal{L}_1^{(2i+1)}\right]\right) \\
&- \sum_{i=0}^{n} \left(2P\left[\mathcal{L}_1^{(2i+1)}\right] E\left[t^{(2i+1)} \mid \mathcal{L}_1^{(2i+1)}\right] + P\left[\mathcal{X}_1^{(2i+1)}\right] E\left[t^{(2i+1)} \mid \mathcal{X}_1^{(2i+1)}\right]\right),
\end{aligned}$$

where the last equality follows from a simple induction and the fact that $E\left[\left(t^{(0)}\right)^2\right] = 0$. $\square$

4.3 Expectation of $r\,t$

Theorem 7. *The expectation $E\left[r^{(x)}\,t^{(x)}\right]$ of $r\,t$ after $x > 1$ steps is given by*

$$E\left[r^{(x)}\,t^{(x)}\right] = -\sum_{i=0}^{\left\lfloor \frac{x}{2} \right\rfloor - 1} \left(\frac{1}{2}P\left[\mathcal{X}_1^{(2i+1)}\right] E\left[r^{(2i+1)} \mid \mathcal{X}_1^{(2i+1)}\right] - P\left[\mathcal{L}_1^{(2i+1)}\right] E\left[r^{(2i+1)} \mid \mathcal{L}_1^{(2i+1)}\right]\right).$$

Proof. Our method here is similar to that of Theorem 6.

We first claim that $E\left[r^{(2n+1)}\,t^{(2n+1)}\right] = E\left[r^{(2n)}\,t^{(2n)}\right]$. To establish this, we fix the values of r and t after $2n$ steps as x and y. Thus, we find that

$$\begin{aligned}
&E\left[r^{(2n+1)}\,t^{(2n+1)} \mid r^{(2n)} = x \wedge t^{(2n)} = y\right] \\
&\qquad = \left(1 - P\left[\mathcal{L}_0^{(2n)}\right]\right) x\,y + P\left[\mathcal{L}_0^{(2n)}\right] \frac{1}{2}\left((x+1)\,y + (x-1)\,y\right) = x\,y,
\end{aligned}$$

so the expectation does not change after $2n$ steps for any values of x and y, as desired.

Therefore, we only need consider $E\left[r^{(2n+2)}\,t^{(2n+2)}\right]$. Fix the values of r and t after $2n+1$

steps as x and y. Considering cases, we see that

$$\begin{aligned}
&E\Big[r^{(2n+2)}\, t^{(2n+2)} \,|r^{(2n+1)} = x \wedge t^{(2n+1)} = y\Big] \\
&\qquad = \left(1 - P\left[\mathcal{X}_1^{(2n+1)}\right] - P\left[\mathcal{X}_2^{(2n+1)}\right] - P\left[\mathcal{L}_1^{(2n+1)}\right] - P\left[\mathcal{L}_2^{(2n+1)}\right]\right) x\, y \\
&\qquad\quad + P\left[\mathcal{X}_1^{(2n+1)}\right] \left(\frac{1}{4}\, x(y-1) + \frac{3}{4}\, x\, y\right) + P\left[\mathcal{X}_2^{(2n+1)}\right] \left(\frac{1}{4}\, x(y+1) + \frac{3}{4}\, x\, y\right) \\
&\qquad\quad + P\left[\mathcal{L}_1^{(2n+1)}\right] \frac{1}{2} \Big(x(y-1) + x\, y\Big) + P\left[\mathcal{L}_2^{(2n+1)}\right] \frac{1}{2} \Big(x(y+1) + x\, y\Big) \\
&\qquad = x\, y - \frac{1}{4}\, P\left[\mathcal{X}_1^{(2n+1)}\right] x + \frac{1}{4}\, P\left[\mathcal{X}_2^{(2n+1)}\right] x - \frac{1}{2}\, P\left[\mathcal{L}_1^{(2n+1)}\right] x + \frac{1}{2}\, P\left[\mathcal{L}_2^{(2n+1)}\right] x.
\end{aligned}$$

Thus, we sum over x and y and apply Bayes' Theorem to find

$$\begin{aligned}
&E\Big[r^{(2n+2)}\, t^{(2n+2)}\Big] \\
&\; = \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} P\left[r^{(2n+1)} = x \wedge t^{(2n+1)} = y\right] E\left[r^{(2n+2)}\, t^{(2n+2)} \mid r^{(2n+1)} = x \wedge t^{(2n+1)} = y\right] \\
&\; = E\left[r^{(2n+1)}\, t^{(2n+1)}\right] - \frac{1}{4}\, P\left[\mathcal{X}_1^{(2n+1)}\right]\, E\left[r^{(2n+1)} \mid \mathcal{X}_1^{(2n+1)}\right] + \frac{1}{4}\, P\left[\mathcal{X}_2^{(2n+1)}\right]\, E\left[r^{(2n+1)} \mid \mathcal{X}_2^{(2n+1)}\right] \\
&\quad - \frac{1}{2}\, P\left[\mathcal{L}_1^{(2n+1)}\right]\, E\left[r^{(2n+1)} \mid \mathcal{L}_1^{(2n+1)}\right] + \frac{1}{2}\, P\left[\mathcal{L}_2^{(2n+1)}\right]\, E\left[r^{(2n+1)} \mid \mathcal{L}_2^{(2n+1)}\right].
\end{aligned}$$

By symmetry, we have $P\left[\mathcal{X}_1^{(2n+1)}\right] = P\left[\mathcal{X}_2^{(2n+1)}\right]$ and $P\left[\mathcal{L}_1^{(2n+1)}\right] = P\left[\mathcal{L}_2^{(2n+1)}\right]$. Furthermore, we know that $E\left[r^{(2n+1)} \mid \mathcal{X}_1^{(2n+1)}\right] = E\left[-r^{(2n+1)} \mid \mathcal{X}_2^{(2n+1)}\right]$ and $E\left[r^{(2n+1)} \mid \mathcal{L}_1^{(2n+1)}\right] = E\left[-r^{(2n+1)} \mid \mathcal{L}_2^{(2n+1)}\right]$, since we can biject walks in the corresponding categories by reflection about the x-axis. Thus, we find that

$$\begin{aligned}
&E\Big[r^{(2n+2)}\, t^{(2n+2)}\Big] \\
&\quad = E\big[r^{(2n)}\, t^{(2n)}\big] - \frac{1}{2}\, P\Big[\mathcal{X}_1^{(2n+1)}\Big]\, E\Big[r^{(2n+1)} \mid \mathcal{X}_1^{(2n+1)}\Big] - P\Big[\mathcal{L}_1^{(2n+1)}\Big]\, E\Big[r^{(2n+1)} \mid \mathcal{L}_1^{(2n+1)}\Big] \\
&\quad = -\sum_{i=0}^{n} \frac{1}{2} P\left[\mathcal{X}_1^{(2n+1)}\right]\, E\left[r^{(2i+1)} \mid \mathcal{X}_1^{(2n+1)}\right] - P\left[\mathcal{L}_1^{(2n+1)}\right]\, E\left[r^{(2i+1)} \mid \mathcal{L}_1^{(2n+1)}\right]
\end{aligned}$$

where again the equality follows from a simple induction and the fact that $E\left[r^{(0)}\, t^{(0)}\right] = 0$. $\square$

4.4 Expectation of c^2

In the previous three sections, we found $E\left[\left(r^{(x)}\right)^2\right]$, $E\left[r^{(x)}\, t^{(x)}\right]$, and $E\left[\left(t^{(x)}\right)^2\right]$. Now, by Theorems 5, 6, and 7, we find that

$$\begin{aligned}
E\left[\left(c^{(x)}\right)^2\right] &= E\left[\left(r^{(x)}+t^{(x)}\right)^2\right] = E\left[\left(r^{(x)}\right)^2\right] + 2E\left[r^{(x)}\, t^{(x)}\right] + E\left[\left(t^{(x)}\right)^2\right] \\
&= 1 + \sum_{i=1}^{\left\lfloor\frac{x-1}{2}\right\rfloor} \frac{\binom{2i}{i}}{2^{2i}}\left(\frac{1}{2}+\frac{\binom{2i}{i}}{2^{2i+1}}\right) + \sum_{i=0}^{\left\lfloor\frac{x}{2}\right\rfloor-1}\left(\frac{1}{2}P\left[\mathcal{X}_1^{(2i+1)}\right] + P\left[\mathcal{L}_1^{(2i+1)}\right]\right) \\
&\quad - \sum_{i=0}^{\left\lfloor\frac{x}{2}\right\rfloor-1}\left(P\left[\mathcal{X}_1^{(2i+1)}\right]\, E\left[t^{(2i+1)} \mid \mathcal{X}_1^{(2i+1)}\right] + 2P\left[\mathcal{L}_1^{(2i+1)}\right]\, E\left[t^{(2i+1)} \mid \mathcal{L}_1^{(2i+1)}\right]\right) \\
&\quad - 2\sum_{i=0}^{\left\lfloor\frac{x}{2}\right\rfloor-1}\left(\frac{1}{2}P\left[\mathcal{X}_1^{(2i+1)}\right]\, E\left[r^{(2i+1)} \mid \mathcal{X}_1^{(2i+1)}\right] + P\left[\mathcal{L}_1^{(2i+1)}\right]\, E\left[r^{(2i+1)} \mid \mathcal{L}_1^{(2i+1)}\right]\right) \\
&= 1 + \sum_{i=1}^{\left\lfloor\frac{x-1}{2}\right\rfloor}\left(\frac{\binom{2i}{i}}{2^{2i+1}} + \frac{\binom{2i}{i}^2}{2^{4i+1}}\right) + \sum_{i=0}^{\left\lfloor\frac{x}{2}\right\rfloor-1}\left(\frac{\binom{2i+1}{i}^2}{2^{4i+3}} + \frac{\binom{2i+1}{i}}{2^{2i+2}}\right) \\
&\quad - \sum_{i=0}^{\left\lfloor\frac{x}{2}\right\rfloor-1}\left(\frac{\binom{2i+1}{i}^2}{2^{4i+2}}\, E\left[c^{(2i+1)} \mid \mathcal{X}_1^{(2i+1)}\right] + 2\frac{\binom{2i+1}{i}}{2^{2i}}\, E\left[c^{(2i+2)} \mid \mathcal{L}_1^{(2i+1)}\right]\right).
\end{aligned}$$

Therefore, we wish to find $E\left[c^{(2i+1)} \mid \mathcal{X}_1^{(2i+1)}\right]$ and $E\left[c^{(2i+1)} \mid \mathcal{L}_1^{(2i+1)}\right]$.

Definition. Let P_i denote the set of all paths of $2i+1$ diagonal steps starting at $(1,0)$ and ending on L_1. For every path p in P_i, let q be the suffix of p that begins at the last intersection of p with the positive x-axis. Denote the multiset of all q by Q_i.

Definition. Let $K(i)$ be the number of paths in Q_i that never touch the negative x-axis. Let $X(i)$ be the number of paths in Q_i whose first move is into the upper half-plane and who touch the negative x-axis.

Theorem 8. *The expectation* $E\left[c^{(2i+1)} \mid \mathcal{L}_1^{(2i+1)}\right]$ *is given by*

$$E\left[c^{(2i+1)} \mid \mathcal{L}_1^{(2i+1)}\right] = \frac{K(i)}{K(i)+2X(i)}.$$

Proof. If we condition on the event that the walk ends on L_0, then the expectation of c is 0, since the walk and its reflection about the x-axis make no net contribution to the expectation. Because c is linear with respect to path concatenation, we only need to compute the expectation of c on walks in Q_i. For the remainder of the proof, q is a walk drawn uniformly from Q_i, t^q and r^q denote the values of t and r at the endpoint of q, and P^+ and P^- denote the upper and lower half planes. Since q never returns to the positive x-axis, $E\left[t^q\right] = 0$. Now, note that r^q is 1 if the first step of q is in P^+, -1 otherwise. By definition, there are $K(i) + X(i)$ walks with first step in P^+. Now, given a walk w with first step in P^-, let the suffix of w beginning at its last contact with the negative x-axis be v. Reflecting v about the x-axis bijects the set of walks w with the walks counted by $X(i)$. Thus, we have

$$E\left[c^{(2i+1)} \mid \mathcal{L}_1^{(2i+1)}\right] = E\left[r^{(2i+1)} \mid \mathcal{L}_1^{(2i+1)}\right] = \frac{K(i) + X(i) - X(i)}{K(i) + 2X(i)} = \frac{K(i)}{K(i) + 2X(i)}. \qquad \square$$

Lemma 9. *The value of $K(i)$ is given by*

$$K(i) = \sum_{j=0}^{i} \sum_{k=0}^{j} \binom{2j}{j+k} \binom{2j}{j} C_{i-j} \sum_{l=k+j-i-1}^{k+i-j} \binom{2i-2j+1}{i-j+1+l-k}$$

Proof. Observe that the paths counted by $K(i)$ touch the positive x-axis for the last time at $(2k+1, 0)$ after $2j$ moves and end at $(2l, 1)$ after another $2i - 2j + 1$ moves without moving below the x-axis for some j, k, and l. There are $\binom{2j}{j+k}\binom{2j}{j}$ and $C_{i-j}\binom{2i-2j+1}{i-j+1+l-k}$ paths that satisfy the first and second conditions, respectively. Summing this value over the range of j, k, and l, we obtain the desired. $\square$

Definition. Define $F(a, b, n)$ to be the number of paths from $(2a+1, 0)$ to $(-2b-1, 0)$ in $2n$ moves that touch the positive x-axis only at their starting points.

Lemma 10. *The value of $F(a, b, n)$ is given by*

$$F(a, b, n) = 4^{n-a-b} \frac{2b+1}{n+a+b+1} \binom{2n-1}{n-a-b-1} \binom{2a}{a} \binom{2b}{b}.$$

Proof. Adopting the notations given in Corollary 16 of [3], we have that

$$
\begin{aligned}
B_{2b+2,0}(z;t) &= \sqrt{\Delta_+(z)}\, S^+_{2b+2,0}(z;t) = \sqrt{\Delta_+(z)} \sum_{k\geq 0} z^{2k}\, S_{2b+2k+2,0}(t) \\
&= \sqrt{1-4t^2\, z^2\, C(4t^2)^2} \sum_{k\geq 0} z^{2k} \binom{2b+2k+2}{b+k+1} t^{2b+2k+2}\, C(4t^2)^{2b+2k+2} \\
&= \left(\sum_{e\geq 0} z^{2e} \left(-\frac{1}{2}\right)_e \frac{(4t^2\, C(4t^2)^2)^e}{e!}\right) \left(\sum_{k\geq 0} z^{2k} \binom{2b+2k+2}{b+k+1} t^{2b+2k+2}\, C(4t^2)^{2b+2k+2}\right),
\end{aligned}
$$

where $\left(-\frac{1}{2}\right)_e$ is the Pochhammer symbol and $C(a)$ is the generating function for the Catalan numbers. Collecting coefficients of z^{2i}, we see that

$$
\begin{aligned}
B_{2b+2,0}(z;t) &= \sum_{i\geq 0} z^{2i} \sum_{j=0}^{i} \left(-\frac{1}{2}\right)_j \frac{(4t^2\, C(4t^2)^2)^j}{j!} \binom{2b+2i-2j+2}{b+i-j+1} t^{2b+2i-2j+2}\, C(4t^2)^{2b+2i-2j+2} \\
&= \sum_{i\geq 0} z^{2i} \sum_{j=0}^{i} -\frac{(2j)!}{(2j-1)\, 2^{2j}\, (j!)} \frac{4^j}{j!} \binom{2b+2i-2j+2}{b+i-j+1} t^{2b+2i+2}\, C(4t^2)^{2b+2i+2}.
\end{aligned}
$$

But $C(a)^n$ is the generating function for the generalized Catalan numbers (Appendix B), so

$$
\begin{aligned}
B_{2b+2,0}(z;t) &= \sum_{i\geq 0} z^{2i} \sum_{j=0}^{i} -\frac{(2j)!}{(2j-1)\,(j!)^2} \binom{2b+2i-2j+2}{b+i-j+1} t^{2b+2i+2} \sum_{k\geq 0} C_k^{2b+2i+2}\, (4t^2)^k \\
&= \sum_{i\geq 0} z^{2i} \sum_{n\geq b+i+1} t^{2n} \sum_{j=0}^{i} -\frac{(2j)!}{(2j-1)\,(j!)^2} \binom{2b+2i-2j+2}{b+i-j+1} C_{n-b-i-1}^{2b+2i+2}\, 4^{n-b-i-1}.
\end{aligned}
$$

Therefore, by the definition of $F(a,b,n)$ and using Corollary 16, we have

$$
F(a,b,n) = [z^{2a}\, t^{2n}] B_{2b+2,0}(z;t) = C_{n-a-b-1}^{2a+2b+2}\, 4^{n-a-b-1} \sum_{x=0}^{a} \frac{-\binom{2x}{x}}{(2x-1)} \binom{2a+2b-2x+2}{a+b-x+1}. \quad (3)
$$

To simplify $F(a,b,n)$, we now establish by induction the identity

$$
\sum_{x=0}^{a} \frac{\binom{2x}{x}}{(2x-1)} \binom{2k-2x}{k-x} = \frac{(2a-2k+1)(a+1)\binom{2a+2}{a+1}\binom{2k-2a-2}{k-a-1}}{k\,(2a+1)} \quad \text{for all } a \leq k. \qquad (4)
$$

Note that the base case $a=0$ holds. Supposing the statement is true for some $a<k$, we have

$$\sum_{x=0}^{a+1} \frac{\binom{2x}{x}}{(2x-1)} \binom{2k-2x}{k-x} = \frac{(2a-2k+1)(a+1)\binom{2a+2}{a+1}\binom{2k-2a-2}{k-a-1}}{k\,(2a+1)} + \frac{\binom{2a+2}{a+1}}{(2a+1)} \binom{2k-2a-2}{k-a-1}$$
$$= \frac{\binom{2a+2}{a+1}\binom{2k-2a-2}{k-a-1}}{k}(a-k+1) = \frac{(2a-2k+3)(a+2)\binom{2a+4}{a+2}\binom{2k-2a-4}{k-a-2}}{k\,(2a+3)},$$

as desired. Letting $k=a+b+1$ in (4) and substituting into (3), we find that

$$F(a,b,n) = 4^{n-a-b-1}\frac{2a+2b+2}{n+a+b+1}\binom{2n-1}{n-a-b-1}\frac{(2b+1)(a+1)\binom{2a+2}{a+1}\binom{2b}{b}}{(a+b+1)\,(2a+1)}$$
$$= 4^{n-a-b}\frac{2b+1}{n+a+b+1}\binom{2n-1}{n-a-b-1}\binom{2a}{a}\binom{2b}{b}. \qquad \square$$

Lemma 11. *The value of $X(i)$ is given by*

$$X(i) = \sum_{j=0}^{i}\sum_{k=0}^{\min\{j,i-j-2\}}\binom{2j}{j+k}\binom{2j}{j}\sum_{l=0}^{\frac{i-j-k-2}{2}}\sum_{m=k+l}^{i-j-l-2}$$
$$F(k,l,m+1)\,C_{i-j-m-1}\sum_{n=1}^{i-j-m-l-1}\binom{2i-2j-2m-1}{i-j-m+n+l}.$$

Proof. Breaking down each path described by $X(i)$ into a path from $(1,0)$ to $(2k+1,0)$ in $2j$ moves, a path from $(2k+1,0)$ to $(-2l-1,0)$ in $2m+2$ moves that avoids the positive x-axis, and a path from $(-2l-1,0)$ to $(2n,1)$ in $2i-2j-2m-1$ moves that stays above the x-axis, we sum over all possible ranges of j, k, l, m, and n to find that

$$X(i) = \sum_{j=0}^{i}\sum_{k=0}^{\min\{j,i-j-2\}}(\#\text{ of paths from }(1,0)\text{ to }(2k+1,0)\text{ in }2j\text{ moves})$$
$$\sum_{l=0}^{\frac{i-j-k-2}{2}}\sum_{m=k+l}^{i-j-l-2}(\#\text{ of paths from }(2k+1,0)\text{ to }(-2l-1,0)\text{ in }2m+2\text{ moves that avoid }L_0)$$
$$\sum_{n=1}^{i-j-m-l-1}(\#\text{ of paths from }(-2l-1,0)\text{ to }(2n,1)\text{ in }2i-2j-2m-1\text{ moves that avoid the }x\text{-axis}),$$

which implies the desired result. $\square$

We now derive the analogous relationships for X_1.

Definition. Let P_i' denote the set of all paths of $2i+1$ diagonal steps starting at $(1,0)$ and ending at X_1. For every path p' in P_i', let q' be the suffix of p' that begins at the last intersection of p' with the positive x-axis. Denote the multiset of all q' by Q_i'.

Definition. Let $K'(i)$ be the number of paths in Q_i' that never touch the negative x-axis. Let $X'(i)$ be the number of paths in Q_i' whose first move is into the upper half-plane and who touch the negative x-axis.

Theorem 12. *The expectation* $E\left[c^{(2i+1)} \mid \mathcal{X}_1^{(2i+1)}\right]$ *is given by*

$$E\left[c^{(2i+1)} \mid \mathcal{X}_1^{(2i+1)}\right] = \frac{K'(i)}{K'(i)+2X'(i)}.$$

Proof. The proof is analogous to that of Theorem 8, with X_1 replacing L_1. □

Lemma 13. *The value of* $K'(i)$ *is given by*

$$K'(i) = \sum_{j=0}^{i}\sum_{k=0}^{j}\binom{2j}{j+k}\binom{2j}{j}C_{i-j}\binom{2i-2j+1}{i-j+1-k}$$

Proof. The proof is similar to that of Lemma 9. □

Lemma 14. *The value of* $X'(i)$ *is given by*

$$X'(i) = \sum_{j=0}^{i}\sum_{k=0}^{\min\{j,i-j-1\}}\binom{2j}{j+k}\binom{2j}{j}\sum_{l=0}^{\frac{i-j-k-1}{2}}\sum_{m=k+l}^{i-j-l-1}F(k,l,m+1)C_{i-j-m-1}\binom{2i-2j-2m-1}{i-j-m+l}.$$

Proof. The proof is analogous to that of Lemma 11. □

Therefore, we have our final explicit value for $E\left[\left(c^{(x)}\right)^2\right]$.

Theorem 15. *The expectation* $E\left[\left(c^{(x)}\right)^2\right]$ *is given by*

$$E\left[\left(c^{(x)}\right)^2\right] = 1 + \sum_{i=1}^{\left\lfloor \frac{x-1}{2} \right\rfloor} \left(\frac{\binom{2i}{i}}{2^{2i+1}} + \frac{\binom{2i}{i}^2}{2^{4i+1}} \right) + \sum_{i=0}^{\left\lfloor \frac{x}{2} \right\rfloor - 1} \left(\frac{\binom{2i+1}{i}^2}{2^{4i+3}} + \frac{\binom{2i+1}{i}}{2^{2i+2}} \right)$$
$$- \sum_{i=0}^{\left\lfloor \frac{x}{2} \right\rfloor - 1} \left(\frac{\binom{2i+1}{i}^2}{2^{4i+2}} \frac{K'(i)}{K'(i) + 2X'(i)} + 2 \frac{\binom{2i+1}{i}}{2^{2i+2}} \frac{K(i)}{K(i) + 2X(i)} \right).$$

Proof. This follows from Theorems 8 and 12. □

Remark. Considering the summations in Theorem 15 asymptotically, we see that the first two summations are $O(\sqrt{n})$, confirming Theorem 4.

5 Conclusions

In this paper, we gave upper and lower bounds of $O(n^{1/4})$ and $\omega(1)$ for the RMS expected winding number after n diagonal steps of a random walk on $\mathbb{Z}^2$ beginning at $(1,0)$ using probabilistic techniques. We then successfully derived an explicit formula for the expected value $E(c^2 \mid x)$ in terms of a binomial sum; we first find the expectation recursively and then exploit a symmetry of random walks to solve the recursion. We also showed that this value is consistent with the bounds given. This result gives us a better understanding of the rotational properties of random walks and thus may be useful in further investigations into this field.

In the future, we hope to generalize this result from the diagonal steps studied in this paper to other categories of steps on the lattice. It would also be interesting to note the effect of the starting point on the final result.

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Appendix A   Expectation of steps for one wind

Theorem 16. *The expected number of steps needed to achieve one wind is infinite.*

Proof. Note that every wind must involve a return to the line $y = x$ and that each step changes the distance to this line by either $\frac{\sqrt{2}}{2}$ or $-\frac{\sqrt{2}}{2}$. We represent this process as a one-dimensional random walk on this distance. It is a well-known result that a one-dimensional random walk requires an infinite expected time to return to its origin [6]. No walk can occur without a return to $y = x$, so $E(\#\text{ of steps} \mid 1\text{ wind})$ is infinite, as desired. □

Appendix B   The Generalized Catalan Numbers

Definition. Let the generalized Catalan number C_n^s be the number of sequences $\{a_i\}_i^n$ with

$$a_i \in \{-1, 1\}\ ,\ \sum_{i=0}^{2n+s} a_i = -s\ ,\ \text{and}\ \sum_{i=0}^{j} a_i > -s \text{ for } j < 2n + s.$$

Remark. Note that $C_n^1 = \frac{1}{n+1}\binom{2n}{n}$ is equal to the ordinary Catalan number C_n, since it counts Catalan sequences of length $2n$ with a -1 appended.

Theorem 17. *The generalized Catalan number C_n^s is given by*

$$C_n^s = \frac{s}{n+s}\binom{2n+s-1}{n}$$

and has generating function

$$f_s(a) = \sum_{n=s}^{\infty} C_n^s\, a^n = C(a)^s,$$

where $C(a) = \frac{1-\sqrt{1-4a}}{2a}$ is the generating function for the Catalan numbers.

Proof. In this proof, we interpret C_n^s as the number of sequences of steps of the form L, $(-1, 0)$, or R, $(1, 0)$, which start at $(s, 0)$ and first reach $(0, 0)$ after $2n + s$ steps. Call such a sequence a (n, s)-sequence. Let the *tail* of a (n, s)-sequence consist of all terms after the first.

We first establish the explicit formula. Consider the (n, s)-sequences with $n > 0$ and $s > 1$ beginning with an L. The tails of these sequences start at $(s-1, 0)$, contain $2n+s-1$ steps, and reach $(0, 0)$ for the first time after $2n + s - 1$ steps. They are thus enumerated by C_n^{s-1}. Now, consider the sequences beginning with an R. They have tails starting at $(s + 1, 0)$, containing $2(n - 1) + s + 1$ steps, and reaching $(0, 0)$ for the first time after $2(n - 1) + s + 1$ steps. Thus, they are enumerated by C_{n-1}^{s+1}. Hence, we have the recurrence relation

$$C_n^s = C_n^{s-1} + C_{n-1}^{s+1} \text{ for } n > 0 \text{ and } s > 1. \tag{5}$$

We now proceed by a double induction on n and then s. Note that $C_0^s = 1 = \frac{s}{0+s}\binom{2\cdot 0+s-1}{0}$, since there is only one possible sequence (all L's), and that $C_n^1 = C_n = \frac{1}{n+1}\binom{2n+1-1}{n}$. Thus, we have established the base cases $s = 1$ and $n = 0$. Now suppose $C_n^s = \frac{s}{n+s}\binom{2n+s-1}{n}$ for all $n < x$. Suppose again that $C_x^s = \frac{s}{x+s}\binom{2x+s-1}{x}$ for all $s < y$. Applying the recurrence in (5), we complete the induction by finding

$$\begin{aligned} C_x^y = C_x^{y-1} + C_{x-1}^{y+1} &= \frac{y-1}{x+y-1}\binom{2x+y-2}{x} + \frac{y+1}{x+y}\binom{2x+y-2}{x-1} \\ &= \frac{(2x+y-2)!}{(x-1)!\,(x+y-1)!}\left[\frac{y-1}{x} + \frac{y+1}{x+y}\right] \\ &= \frac{(2x+y-2)!}{(x-1)!\,(x+y-1)!}\,\frac{y\,(2x+y-1)}{x(x+y)} \\ &= \frac{y}{x+y}\binom{2x+y-1}{x}. \end{aligned}$$

We now establish the generating function. First, note that any $(n, 1)$-sequence is a Catalan sequence with an L appended, so we can biject them. Thus, $f_1(a) = C(a)$. Now, divide an (n, s)-sequence into s subsequences that extend between the first times $(k, 0)$ and $(k - 1, 0)$ are reached for $s \geq k \geq 1$. Each of these subsequences satisfies the properties of a $(n, 1)$-sequence when shifted to the left by $k - 1$ units, so each (n, s)-sequence is a concatenation of s $(n, 1)$-sequences. Hence, we see that $C(a)^s$, the product of the generating functions of the $(n, 1)$-sequences, is the generating function for C_n^s. □

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