Polyominoes

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Abstract

Using combinatorial methods, we count various classes of polyominoes with certain restrictions, e.g. parallelogram polyominoes by minimal bounding rectangle.

1 Introduction

A polyomino can be thought of as a union of 1×1 lattice squares with connected interior. Familiar examples of polyominoes are dominoes, rectangles made of two 1×1 boxes, and Tetris pieces, figures made of four 1×1 boxes. We are interested in counting polyominoes. However, counting the total number of polyominoes of a given area is an unsolved and very difficult problem; likewise, enumerating all polyominoes of a given perimeter or in a given bounding rectangle are relatively intractable problems. Instead of attempting to count all polyominoes with a given parameter limited, we classified certain sets of polyominoes and counted the number of polyominoes in a given category. Classes of polyominoes include rectangles, skyline polyominoes, vertically convex polyominoes, parallelogram polyominoes, and directed convex polyominoes. Sometimes, it was helpful to add symmetry as an additional constraint. We counted the polyominoes in a given class, restricting different parameters in each case. The various ways to count polyominoes are by minimal bounding rectangle, by area, and by perimeter.

2 Definitions and Important Concepts

A *polyomino* is a lattice figure in which the bounded region, not including the boundary, is connected.

A *vertically convex* polyomino is a polyomino whose intersection with any vertical line is at most one line segment.

A *horizontally convex* polyomino is a polyomino whose intersection with any horizontal line is at most one line segment.

We say a polyomino is *convex* if it is both vertically and horizontally convex.

We count polyominoes by area, perimeter, and minimal bounding rectangle. *Area* and *perimeter* are used in the common senses. By the *minimal bounding rectangle* of a polyomino, we mean the rectangle with the smallest width and height that can contain the polyomino.

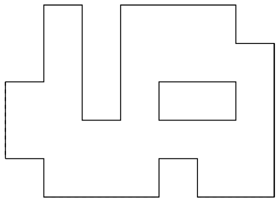

Figure 1: A polyomino

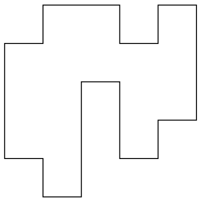

Figure 2: A vertically convex polyomino

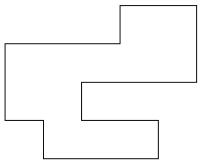

Figure 3: A horizontally convex polyomino

In counting polyominoes, we will need to consider Dyck paths and Motzkin paths.

A *Dyck path* of length n is a sequence of n northeast and n southeast steps such that the path begins and ends on the x-axis and never dips below it.

The number of Dyck paths of length n is equal to the n^{th} Catalan number, C_n, where $C_n = \frac{1}{n+1}\binom{2n}{n}$.

A *Motzkin path* is a Dyck path which may contain horizontal steps. A *2-colored Motzkin path* is simply a Motzkin path in which each horizontal element is colored one of two colors.

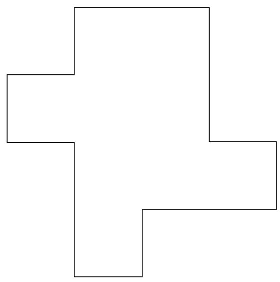

Figure 4: A convex polyomino

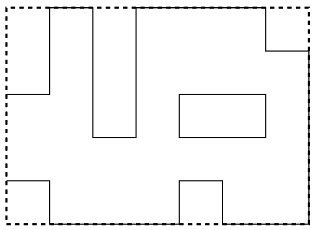

Figure 5: A polyomino with its bounding rectangle

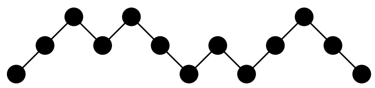

Figure 6: A Dyck path

In this paper, we will frequently be using paths of up and right steps to define the boundary of a polyomino. Let us first begin by introducing the technique of counting paths composed of up and right steps from a point (a, b) to (c, d) with $c - a > 0$ and $d - b > 0$

There are $c - a$ up steps and $d - b$ right steps in this path. Since these steps can be arranged in any order, there are $\binom{c-a+d-b}{c-a}$ paths from (a, b) to (c, d). To define our polyomino's boundary, we will be considering pairs of paths. Because we do not want these paths to touch in order for the interior

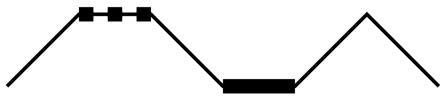

Figure 7: A 2-colored Motzkin path. The bold horizontal line represents one color and the dotted horizontal line represents another.

to be connected, we will be making use of the Reflection Principle to count the pairs of paths which do touch. Let us consider a path from (a, b) to (c, d) and one from (w, x) to (y, z). We require that (a, b), (c, d), (w, x), and (y, z) lie on a rectangle with none of the points on the vertices of the rectangle and with (a, b) on the left side, (c, d) on the top side, (y, z) on the right side, (w, x) on the bottom side, $b < z$, and $w < c$.

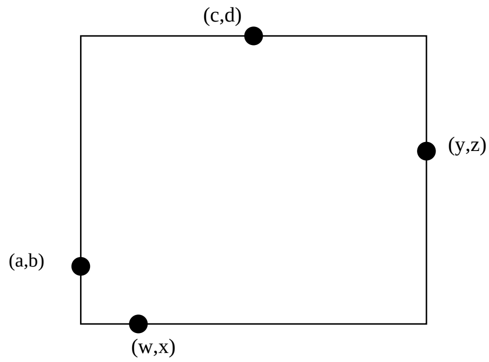

Figure 8: The points (a, b), (c, d), (w, x), and (y, z),with certain restrictions, lie on a rectangle.

We can create a bijection between pairs of paths, one from (a, b) to (c, d) and the other from (w, x) to (y, z) that touch and pairs of paths, one from (a, b) to (y, z) and one from (w, x) to (c, d) by taking our initial pair of paths and letting one follow the steps of the other after the lowest leftmost point of contact.

There are $\binom{c+d-w-x}{c-w}$ paths from (w, x) to (c, d) and $\binom{y+z-a-b}{y-a}$ paths from (a, b) to (y, z) and thus, $\binom{c+d-w-x}{c-w}\binom{y+z-a-b}{y-a}$ pairs of paths, where one goes

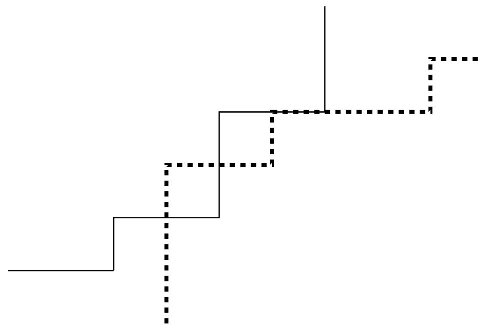

Figure 9: An initial pair of paths. The dotted line goes from (w, x) to (y, z) and the solid line goes from (a, b) to (c, d)

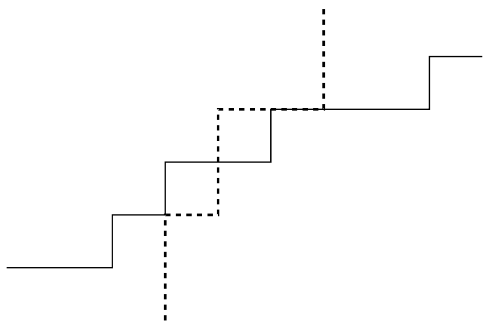

Figure 10: The paths after performing switch at lowest and leftmost intersection point

from (a, b) to (c, d) and the other from (w, x) to (y, z), which touch.

3 Rectangles

A *rectangle* is a rectangular polyomino in the common sense of the word.

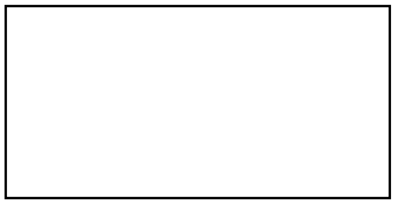

Figure 11: A rectangle

3.1 By Perimeter

Theorem 3.1.1. *The number of rectangular polyominoes with perimeter* $2p$ *is* $p-1$.

Proof. Let our rectangle have height h and width w. Then $h+w=p$. Because p is fixed, once we choose h, w is determined. $1 \leq h \leq p-1$ because h and w must be positive integers. Thus, there are $p-1$ rectangles with perimeter $2p$. □

3.2 By Area

Theorem 3.2.1. *The number of rectangular polyominoes of area* $A = {p_1}^{k_1}{p_2}^{k_2}\ldots{p_n}^{k_n}$ *is* $(k_1+1)(k_2+1)\ldots(k_n+1)$.

Proof. If the rectangle has height h and width w, then $hw = A$. Because A is fixed, once we choose h, w is determined. There are $(k_1+1)(k_2+1)\ldots(k_n+1)$ ways to choose h because we can choose anywhere from 0 to k_i factors of p_i for $1 \leq i \leq n$. □

4 Rectangles with Holes

A *rectangle with a hole* is a rectangle with exactly one rectangular hole. Its bounding rectangle $w_1 \times h_1$ must have $w_1, h_1 \geq 3$ because otherwise there can be no inner hole, as a hole must have length and height ≥ 1 and there must also be at least 1 lattice square in the rectangle on each side of the hole. For the following proofs, let $w_2 \times h_2$ be the dimensions of the hole.

4.1 By Minimal Bounding Rectangle

Theorem 4.1.1. *The number of rectangles with holes given minimal bounding rectangle and fixed hole dimensions is* $(w_1 - w_2 - 1)(h_1 - h_2 - 1)$.

Proof. Let us consider rectangles with holes with fixed outer dimensions $w_1 \times h_1$ and a hole of fixed size $w_2 \times h_2$. We can place our rectangle on the plane so that the lower left point is located at the origin and the upper right corner is at the point (w_1, h_1). The left boundary of the hole cannot coincide with the left boundary of the outer rectangle, so the hole's left boundary should lie on, or to the right of, the line $x = 1$. Moreover, if our left boundary is located on, or to the right of, the line $x = w_1 - w_2$, the right boundary of the hole will coincide with or lie to the right of the right boundary of the rectangle, which is not allowed. Thus, there are $w_1 - w_2 - 1$ possible locations for the left boundary of the hole.

Similarly, there are $h_1 - h_2 - 1$ possible locations for the lower boundary of the hole. Since the hole's dimensions are fixed, the right and upper boundaries are determined once the left and lower boundaries have been chosen, so there must be $(w_1 - w_2 - 1)(h_1 - h_2 - 1)$ rectangles with holes given fixed dimensions for the rectangle and for the hole. □

Theorem 4.1.2. *The number of rectangles with holes given minimal bounding rectangle and variable hole dimensions is* $\binom{w_1-1}{2}\binom{h_1-1}{2}$.

Proof. Now let us consider rectangles with a hole of variable size. Consider the left and right boundaries for the hole. These boundaries must be distinct and there are w_1+1 possible locations for them. However, since the hole must remain in the interior of the rectangle, its boundaries cannot coincide with the rectangle's boundaries. This leaves $(w_1+1)-2 = w_1-1$ possible locations for the left and right boundaries of the hole. Thus we have $\binom{w_1-1}{2}$ possible combinations of left and right boundaries. Similarly, there are $\binom{h_1-1}{2}$ possible combinations of lower and upper boundaries for the hole, so in all there are $\binom{w_1-1}{2}\binom{h_1-1}{2}$ rectangles with holes given minimal bounding rectangle $w_1 \times h_1$. □

4.2 By Perimeter

We begin by introducing a new notation for rectangles with holes.

Proposition 4.2.1. *Each rectangle with a hole can be uniquely represented as a 6-tuple of increasing positive integers $S = (a, b, c, d, e, f)$ where $2f$ is the perimeter of the rectangle.*

Proof. The ordered set S can be thought of as points on a numberline from 0 to f. At the point c, let the numberline bend perpendicularly up and complete a rectangle by treating the bent numberline as the lower and right edges of the rectangle. Thus we have a $c \times (f - c)$ rectangle. Let the left and right boundaries of the hole lie above a and b respectively. Let the lower and upper boundaries of the hole lie to the left of d and e respectively. Thus there is a bijection between rectangles with holes and such 6-tuples S, as any given ordered set S produces a unique rectangle with a hole and by projecting the hole boundaries to the lower and right edges of our rectangle, we can recover our initial 6-tuple S. □

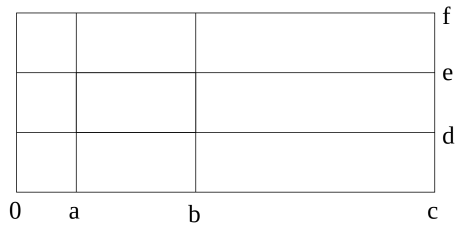

Figure 12: The bent numberline representation of a rectangle with a hole

Theorem 4.2.1. *The number of rectangles with holes given perimeter $2p_1$ and variable hole dimensions is $\binom{p_1-1}{5}$.*

Proof. Now let us consider a rectangle of perimeter $2p_1$ with a variable hole. Given the new representation for a rectangle with a hole from Proposition 4.2.1, $S = (a, b, c, d, e, f)$. Because $f = p_1$, f is fixed. However, a, b, c, d, e remain to be chosen in a manner such that $0 < a < b < c < d < e < p_1$. Then there are $\binom{p_1-1}{5}$ possible ordered sets (a, b, c, d, e) since there is only one way to order the 5 chosen numbers so that they are increasing. Thus, there are $\binom{p_1-1}{5}$ rectangles with perimeter $2p_1$ which contain exactly one rectangular hole. □

Theorem 4.2.2. *The number of rectangles with holes given perimeter $2p_1$ and fixed hole dimensions is $\binom{p_1-p_2-1}{3}$ where $2p_2$ is the perimeter of the hole.*

Proof. Let us consider a rectangle of perimeter $2p_1$ with a hole of dimensions $w_2 \times h_2$ so that $p_2 = w_2 + h_2$. Let our rectangle be represented by the 6-tuple $S = (a, b, c, d, e, f)$. Then $f = p_1$, $(e - d) = h_2$ and $(b - a) = w_2$. Thus knowing the ordered triplet (b, c, d) determines the rest of the rectangle because f is fixed and a, e are respectively determined by b, d. Because $b - a = w_2$ and a is greater than 0 we have that b is strictly greater than w_2. Likewise, $e - d = h_2$ and e is less than f tells us that d is strictly less than $f - h_2$. Then $w_2 < b < c < d < p_1 - h_2$. There are $\binom{p_1-h_2-w_2-1}{3}$, or simply $\binom{p_1-p_2-1}{3}$ possible triplets (b, c, d). Thus there are $\binom{p_1-p_2-1}{3}$ rectangles with perimeter $2p_1$ and fixed hole $w_2 \times h_2$. □

Theorem 4.2.3. *The number of rectangles with holes given perimeter* $2p_1$ *and perimeter of hole* $2p_2$ *is* $(p_2 - 1)\binom{p_1-p_2-1}{3}$.

Proof. We already know from Theorem 4.2.2 that there are $\binom{p_1-p_2-1}{3}$ rectangles with perimeter $2p_1$ and fixed hole $w_2 \times h_2$ such that $p_2 = w_2 + h_2$. Note that in this formula, the number of polyominoes does not depend on the values of w_2 or h_2 but only on the value of their sum p_2. Given p_2, there are $p_2 - 1$ choices for w_2 and h_2 since $0 < w_2, h_2 < p_2$. Thus there are $(p_2 - 1)\binom{p_1-p_2-1}{3}$ rectangles with perimeter $2p_1$ containing a hole of perimeter $2p_2$. □

5 Skyline Polyominoes

A *skyline polyomino* is a vertically convex polyomino consisting of a set of vertical columns that are bottom justified.

5.1 By Area

Proposition 5.1.1. *The number of skyline polyominoes of area* A *and width* w *is* $\left(\!\binom{w}{A-w}\!\right) = \binom{A-1}{w-1}$.

Proof. There are w columns into which we can put A units of area. After we put 1 unit into each of the w columns, so that each column is not empty, we can distribute the remaining $A - w$ units into w columns *with repetition* in $\left(\!\binom{w}{A-w}\!\right) = \binom{A-1}{w-1}$ ways. □

Proposition 5.1.2. *The number of skyline polyominoes of area* A *is* 2^{A-1}.

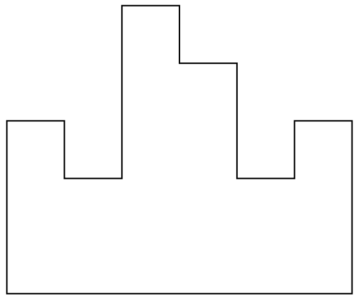

Figure 13: A skyline polyomino

Proof. Each skyline polyomino of area A will have w columns, for some w in the range 1 to A. Therefore, there are $\sum_{w=1}^{A} \binom{A-1}{w-1} = 2^{A-1}$ skyline polyominoes of area A.
We can encode each skyline polyomino uniquely as a sequence of natural number(s) x_i that add up to A, where x_i corresponds to the height of the i^{th} column. It suffices to count the number of such sequences. Consider the arithmetic expression $A = 1+1+1+\ldots+1+1+1$, a line of A 1's separated by $(A-1)$ +'s. We can turn each of the +'s 'ON' (perform the addition, and replace $1+1$ with 2, etc.) or leave it 'OFF' (leave the + sign as is). There are two configurations for each of the +'s, so there are 2^{A-1} total configurations for the $(A-1)$ +'s. Thus, there are 2^{A-1} desired sequences. □

5.2 By Minimal Bounding Rectangle

Proposition 5.2.1. *The number of skyline polyominoes of bounding rectangle $w \times h$ is $h^w - (h-1)^w$.*

Proof. Any skyline polyomino that can be contained in a $w \times h$ rectangle can have a height from 1 to h in each column. Thus, there are h^w skyline polyominoes that can be contained in, but are not necessarily bounded by, a $w \times h$ rectangle. Similarly, there are $(h-1)^w$ skyline polyominoes that can be contained in a $w \times (h-1)$ rectangle. The number of skyline polyominoes bounded by a $w \times h$ rectangle is the number of skyline polyominoes that can be contained in a $w \times h$ rectangle minus the number of skyline polyominoes that can be contained in a $w \times (h-1)$ rectangle. Thus, there are $h^w - (h-1)^w$

skyline polyominoes of bounding rectangle $w \times h$. □

6 Vertically Convex Polyominoes

6.1 By Minimal Bounding Rectangle: A Special Case

The special case we will consider is vertically convex polyominoes with at least one solid row going across a minimal bounding rectangle with dimensions $w \times h$.

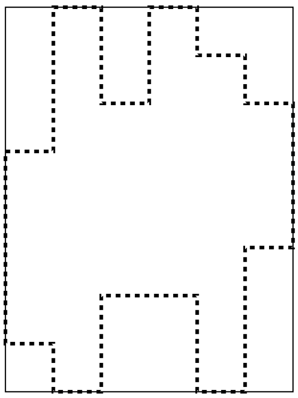

Figure 14: A vertically convex polyomino with at least one solid row going across the minimal bounding rectangle

Theorem 6.1.1. *The number of vertically convex polyominoes with at least one solid row going across a $w \times h$ minimal bounding rectangle is*

$$2(h^w-2(h-1)^w+(h-2)^w)+\sum_{i=1}^{h-1}[(h-i+1)^w-2(h-i)^w+(h-i-1)^w][i^w-(i-1)^w]$$

Proof. First let us consider the number of skyline polyominoes with exactly one solid row at the bottom.

Lemma 6.1.1.1. *There are $h^w - 2(h-1)^w + (h-2)^w$ such polyominoes.*

Proof. By Proposition 5.2.1, there are $h^w-(h-1)^w$ total skyline polyominoes in a $w\times h$ bounding rectangle. Any skyline polyomino with more than one solid row at the bottom can be thought of as a skyline polyomino with a $w\times(h-1)$ bounding rectangle because we can simply remove the bottom row. Thus, there are $h^w-(h-1)^w-(h-1)^w+(h-2)^w = h^w-2(h-1)^w+(h-2)^w$ skyline polyominoes with exactly one solid row at the bottom. □

Now we may subdivide the bounding rectangle into two rectangles, each of width w, whose union is the entire rectangle, and whose intersection is a $w\times 1$ rectangle. Let the bottom rectangle have height $h-i+1$ and the top one have height i. We can generate our vertically convex polyomino by placing a skyline polyomino in the top rectangle, and placing a skyline polyomino which has been reflected over a horizontal line in the bottom rectangle.

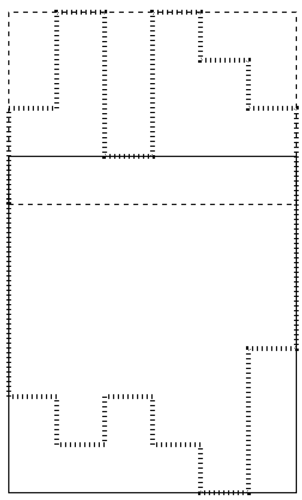

Figure 15: The dotted rectangle is the top bounding rectangle and the solid rectangle is the bottom bounding rectangle.

To avoid double counting, we insist on the condition that our top rectangle contain a skyline polyomino with only one solid row, while our bottom rectangle can contain any skyline polyomino. Then by Lemma 6.1.1.1 there are $(h-i+1)^w-2(h-i)^w+(h-i-1)^w$ skyline polyominoes with the bottom rectangle as its bounding rectangle and only one solid row. By Proposition 5.2.1, there are $i^w-(i-1)^w$ general skyline polyominoes with the top rectangle as its minimal bounding rectangle. So we would think that there are

$\sum_{i=1}^{h-1}[(h-i+1)^w - 2(h-i)^w + (h-i-1)^w][i^w - (i-1)^w]$ vertically convex polyominoes satisfying our conditions. However, this counts neither the skyline polyomino with exactly one solid row at the bottom and $w \times h$ minimal bounding rectangle nor the "upside-down" skyline polyomino with exactly one solid row at the top and $w \times h$ minimal bounding rectangle. There are $h^w - 2(h-1)^w + (h-2)^2$ of each of these types by Lemma 6.1.1.1, so there are

$$2(h^w-2(h-1)^w+(h-2)^w)+\sum_{i=1}^{h-1}[(h-i+1)^w-2(h-i)^w+(h-i-1)^w][i^w-(i-1)^w]$$

vertically convex polyominoes with at least one solid row going across the $w \times h$ bounding rectangle. □

7 Ferrers Polyominoes

A *Ferrers polyomino* is a polyomino that is left and bottom justified. Additionally, the rest of the boundary of each Ferrers polyomino must consist of a sequence of right and down steps starting from the upper left corner and ending at the lower right corner.
Note: The sequence must begin with a right step and end with a down step.

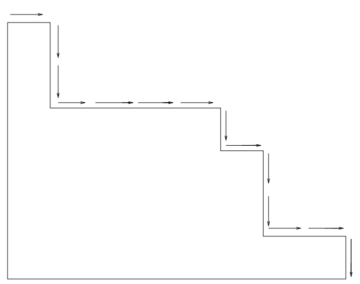

Figure 16: A Ferrers polyomino

7.1 By Minimal Bounding Rectangle

Theorem 7.1.1. *The number of Ferrers polyominoes given minimal bounding rectangle w by h is* $\binom{w+h-2}{h-1}$.

Proof. Given the $w \times h$ minimal bounding rectangle, we know that there must be w steps right and h steps down from the upper left corner of the polyomino to its lower right corner. Since the sequence must begin with a right step and end with a down step, we need only consider the directions of the remaining $(w-1)+(h-1)$ steps. By choosing the locations of the $h-1$ down steps, the $w-1$ right steps are also determined. Thus there are $\binom{w+h-2}{h-1}$ Ferrers polyominoes with minimal bounding rectangle $w \times h$. □

7.2 By Perimeter

Theorem 7.2.1. *The number of Ferrers polyominoes given perimeter* $2p$ *is* 2^{p-2}.

Proof. Given perimeter $2p$, we know that there are in total p steps right and down in the determining sequence for the Ferrers polyomino. Since only the first and last steps are fixed, there remain $p-2$ steps that can be either right or down. Thus there are 2^{p-2} possible sequences of right and down steps, so there are 2^{p-2} Ferrers polyominoes with perimeter $2p$. □

A symmetric Ferrers polyomino can be determined by half of the normal determining arrow sequence, as the second half of the sequence is symmetric to the first half. Additionally, the minimal bounding rectangle of any symmetric polyomino must be symmetric over its diagonal, so it is a square.

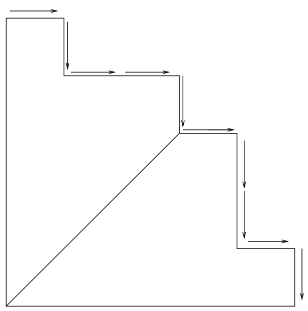

Figure 17: A symmetric Ferrers polyomino

Theorem 7.2.2. *The number of symmetric Ferrers polyominoes given perimeter* $4m$ *(minimal bounding rectangle* $m \times m$*) is* 2^{m-1}.

Proof. We consider a symmetric Ferrers polyomino with minimal bounding rectangle $m \times m$ to be a Ferrers polyomino with perimeter $4m$. Then the polyomino is determined by a sequence of $2m$ right and down steps. Being symmetric the first half of the sequence determines the rest, so we only need to consider the directions of the first $\frac{2m}{2} = m$ steps. Since the first and last steps are fixed and the first step is the only one of the two that is in the first half of the sequence, we need only consider $m-1$ steps. There are 2^{m-1} sequences of directions for these steps, so there are 2^{m-1} symmetric Ferrers polyominoes with minimal bounding rectangle $m \times m$ and perimeter $4m$. □

7.3 By Area

Theorem 7.3.1. *The number of Ferrers polyominoes with area A is $\pi(A)$, where $\pi(A)$ denotes the number of partitions of A.*

Proof. Each partition $P = p_1 p_2 \cdots p_n$ of $A = p_1 + p_2 + \cdots + p_n$ where $p_i \geq p_{1+i}$ corresponds to a unique Ferrers polyomino. We let p_i correspond to the length of the i^{th} row for $1 \leq i \leq n$. Because each row must be left justified, the row lengths completely determine the polyomino. □

8 Stack Polyominoes

A *stack polyomino* is a horizontally convex skyline polyomino.

8.1 By Minimal Bounding Rectangle

Proposition 8.1.1. *The number of stack polyominoes of bounding rectangle $w \times h$ is $\binom{2h+w-3}{2h-2} = \left(\!\binom{w}{2h-2}\!\right)$.*

Proof. Each stack of bounding rectangle $w \times h$ can be described by a lattice path, such that

1. the path begins with an $\uparrow$ and ends with a $\downarrow$ (or else the polyomino is invalid)

2. there are w right-steps ($\rightarrow$), h up-steps ($\uparrow$), and h down-steps ($\downarrow$) (since a stack is unimodal)

3. all of the $\uparrow$'s come before the $\downarrow$'s (since a stack is unimodal)

4. the last $\uparrow$ must be separated from the first $\downarrow$ by a $\rightarrow$ (or else the polyomino is also invalid)

So, it suffices to count the number of such paths. Set the first step to be $\uparrow$, the last step to be $\downarrow$, and constrain a $\rightarrow$ to follow the last $\uparrow$. We are left with $(w-1)$ $\rightarrow$'s and $(2h-2)$ $\updownarrow$'s (both $\uparrow$'s and $\downarrow$'s) to work with. There are $\binom{2h+w-3}{2h-2}$ ways to permute the $(w-1)$ $\rightarrow$'s and $(2h-2)$ $\updownarrow$'s. Letting the first $(h-1)$ $\updownarrow$'s be $\uparrow$'s and the last $(h-1)$ $\updownarrow$'s be $\downarrow$'s will give us a path as desired. Thus, there are $\binom{2h+w-3}{2h-2}$ paths and $\binom{2h+w-3}{2h-2}$ stack polyominoes of bounding rectangle $w \times h$. $\square$

8.2 By Perimeter

Theorem 8.2.1. *There are F_{p-3} stack polyominoes of perimeter p, where $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}\ \forall\ n \geq 2$.*

Proof. Let S_{2n} be the number of stacks with perimeter $2n$.

Lemma 8.2.1.1. $S_{2n} = S_{2n-2} + 2S_{2n-4} + 3S_{2n-6} + \cdots + (n-2)S_4 + 1.$

Proof. Consider the number of stacks with perimeter $2n$, and height ≥ 2. There are $S_{2n} - 1$ of these (all but the simple horizontal stack).

We can consider building such a stack from a smaller stack by adding a new row to the bottom.

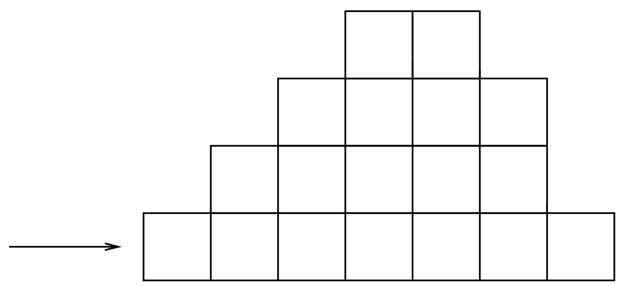

Figure 18

Adding a row equal in width to the row above it raises the perimeter by 2, and can be added in one way. In general, adding a row with i blocks more than the row above it adds $2(i+1)$ to the perimeter, and can be added in $(i+1)$ ways.

So,

$$S_{2n} - 1 = S_{2n-2} + 2S_{2n-4} + \cdots + (n-2)S_4.$$

The result follows by adding 1 to both sides. □

We will make use of the combinatorial properties of the Fibonacci numbers.

Define f_n to be the number of tilings of a horizontal strip of length n with squares and dominoes.

Lemma 8.2.1.2. $f_n = F_{n+1}$

Proof. We proceed by induction on n. Note that $f_0 = 1 = F_1$, and $f_1 = 1 = F_2$, by inspection, satisfying the base cases.

If the first cell is occupied by a square, then there are f_{n-1} ways to tile the rest of the strip.

If the first cell is occupied by a domino, there are f_{n-2} ways to tile the rest of the strip.

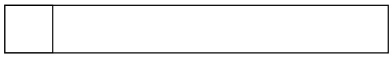

Figure 19: A square occupies the first cell

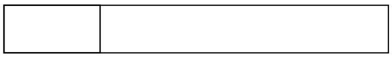

Figure 20: A domino occupies the first cell

Therefore, $f_n = f_{n-1} + f_{n-2} = F_n + F_{n-1} = F_{n+1}$, as desired. □

Lemma 8.2.1.3. $f_{2n} = f_{2n-2} + 2f_{2n-4} + 3f_{2n-6} + \cdots + nf_0 + 1$

Proof. Consider a horizontal strip of length $2n$ that we wish to tile with squares and dominoes. One way to count the number of tilings is f_{2n}, by definition.

Now, either no squares are used to tile the strip, which can happen in one way, or some squares are used. If some squares are used, the number of squares must be even since the length of the strip is even. Therefore, there are at least two squares.

We will count by the position of the second square from the left. Let us label the strip with the numbers $1, 2, 3, \ldots, n$. It is clear the second square must be on an even numbered spot, since the used pieces up to that point, including itself, include 2 squares and some dominoes. So, suppose the second square goes on $2i$, for some $i \geq 1$.

Figure 21: An n-board with shaded second square

Now consider where we can place the first square. Picking this spot will determine the tiling up to spot $2i$. Note for example that the first square cannot go on spot 2, since then spot 1 would have to contain a domino, but this would overlap with the square on spot 2. Likewise, the first square cannot be on any even numbered spot. Therefore, it must be on one of $1, 3, 5, \ldots, (2i-1)$. This means we have i choices for the square.

After the tiling of the first $2i$ squares has been determined, there are $(2n-2i)$ squares left that can be tiled in f_{2n-2i} ways.

Therefore,

$$\begin{aligned} f_{2n} &= \sum_{i=1}^{n} i f_{2n-2i} + 1 \\ &= f_{2n-2} + 2f_{2n-4} + \cdots + nf_0 + 1, \end{aligned}$$

as desired. □

We now show

$$S_{2n} = f_{2n-4} \; \forall \; n \geq 2.$$

The proof proceeds by induction on n. Note that if $n = 2$, $S_4 = 1 = f_0$.

Otherwise,

$$\begin{aligned} S_{2n} &= S_{2n-2} + 2S_{2n-4} + \cdots + (n-2)S_4 + 1 \text{ (by Lemma 8.2.1.1)} \\ &= f_{2n-6} + 2f_{2n-8} + \cdots + (n-2)f_0 + 1 \text{ (by inductive hypothesis)} \\ &= f_{2n-4}, \text{ (by Lemma 8.2.1.3)} \end{aligned}$$

as desired.

Now we have by Lemma 8.2.1.2 that $f_{2n-4} = F_{2n-3}$, so it follows that $S_{2n} = f_{2n-4} = F_{2n-3}$. Note that the perimeter of a stack must be even, so if we let $p = 2n$, $S_p = F_{p-3}$, which was what we wanted to prove.

□

9 Parallelogram Polyominoes

Parallelogram polyominoes are polyominoes bounded by two paths of equal length, each composed of up and right steps, such that the two paths intersect one another at only their beginnings and ends.

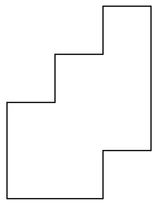

Figure 22: A Parallelogram Polyomino

9.1 By Minimal Bounding Rectangle

Theorem 9.1.1. *The number of parallelogram polyominoes within a $w \times h$ rectangle is* $\binom{h+w-2}{h-1}^2 - \binom{h+w-2}{h}\binom{h+w-2}{w}$

Proof. We can consider each parallelogram polyomino to be a defined by the union of two paths of right and up steps from $(0,0)$ to (w,h), such that the two paths only touch at the points $(0,0)$ and (w,h). Thus, one of the paths must go through $(1,0)$ and the other through $(0,1)$. Moreover, the path through $(1,0)$ must go through $(w,h-1)$ and the path through $(0,1)$ must go through $(w-1,h)$. Then we can say that the number of parallelogram polyominoes with the $w \times h$ bounding rectangle is the total number of pairs of paths, where one goes from $(1,0)$ to $(w,h-1)$ and the other from $(0,1)$ to $(w-1,h)$, minus those pairs of paths which touch one another.

The number of paths from $(1,0)$ to $(w,h-1)$ is $\binom{w+h-2}{h-1}$ as there are $w+h-2$ steps to take, of which exactly $h-1$ can be steps up. Likewise, the number of paths from $(0,1)$ to $(w-1,h)$ is also $\binom{h+w-2}{h-1}$, so our total number of pairs of paths is $\binom{h+w-2}{h-1}^2$.

Here we can make use of the Reflection Principle discussed in *Definitions and Important Concepts* to count those pairs of paths which touch. The number of pairs of paths, one from $(0,1)$ to $(w-1,h)$ and another from $(1,0)$ to $(h,w-1)$, which touch is equal to the number of pairs of paths where one goes from $(1,0)$ to $(w-1,h)$ and the other from $(0,1)$ to $(h,w-1)$. The number of these pairs of paths is $\binom{h+w-2}{h}\binom{h+w-2}{w}$.

Thus the number of pairs of paths, one from $(0,1)$ to $(w-1,h)$ and the other from $(1,0)$ to $(h,w-1)$ which do not touch, and hence the number of parallelogram polyominoes with a $w \times h$ bounding rectangle is $\binom{h+w-2}{h-1}^2 - \binom{h+w-2}{h}\binom{h+w-2}{w}$.

□

9.2 By Perimeter

Theorem 9.2.1. *The number of parallelogram polyominoes with perimeter $2p$ is C_{2p-1}.*

Proof. Define two sequences A_n and B_n as follows:

A_n = number of parallelogram polyominoes of perimeter $2n$,
B_n = number of 2-colored Motzkin paths of length n

Lemma 9.2.1.1. $B_n = C_{n+1}$

Proof. We provide a bijection between 2-colored Motzkin paths of length n and Dyck paths of length $(n+1)$. Since there are C_{n+1} Dyck paths of length $(n+1)$, it follows that there are C_{n+1} Motzkin paths of length n.

Suppose the horizontal steps in the 2-colored Motzkin paths are colored red and blue. We provide a constructive bijection from a Motzkin path of length n to a Dyck path of length $(n+1)$.

Let the first and last steps of this Dyck path be an up step and a down step, respectively. Recall that a Dyck path of length $(n+1)$ has $(2n+2)$ steps by definition, so we must fill the remaining $2n$ internal segments of the path.

Let an up step in the Motzkin path correspond to two consecutive up steps in the Dyck path, and a down step in the Motzkin path correspond to two consecutive down steps in the Dyck path.

Furthermore, let a red horizontal step in the Motzkin path correspond to an up step followed by a down step, while a blue horizontal step corresponds to a down step followed by an up step.

It is clear that if the Motzkin path stays above or on the x-axis, then the corresponding Dyck path stays on or above the x-axis. Furthermore, from any Dyck path we can construct the corresponding Motzkin path, and

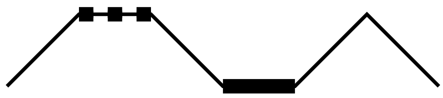

Figure 23: A 2-colored Motzkin path. The dotted horizontal line corresponds to a blue step and the solid horizontal line corresponds to a red step.

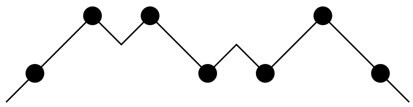

Figure 24: The corresponding Dyck Path

it must definitely stay on or above the x-axis. Therefore, this mapping is a bijection and the resulting equality follows. □

We claim $A_n = B_{n-1}$.

We consider the parallelogram polyomino to be a set of two paths, an upper and a lower path, which intersect only at the beginning and end of the polyomino. We also add the restriction that each path consists of only up and right moves.

The first and last moves of each path are fixed if the figure is to be a valid parallelogram polyomino. Therefore, we consider only the internal parts of the path. The paths start out at a distance of 1 from each other. The distance must remain positive for the paths not to intersect.

If the top path moves right and the bottom path moves up, the distance between the paths is reduced by one. If both paths move up or both move right, then the distance is preserved. If the top path moves up and the bottom moves right, the distance is increased by 1.

We can model these interactions with a 2-colored Motzkin path. If the distance decreases between the paths, which can happen in one way, then we model this with a down step. If the distance increases we model this with an up step. Otherwise, the distance between the two paths can be fixed by one of two moves, which we model as either a blue or red horizontal step.

Therefore,

$$A_n = B_{n-2} = C_{n-1},$$

as desired. □

9.3 With Diagonal Symmetry

Theorem 9.3.1. *The number of parallelogram polyominoes bounded by an $n \times n$ rectangle with symmetry across the line $y = x$ is C_{n-1}.*

Proof. Since our polyomino must have diagonal symmetry, we need only consider a path of up and right steps from $(0, 0)$ to (n, n) to mark half the border of the polyomino, then reflect the path across the line $y = x$ to obtain the other half of the border. We will call the path that begins by taking a step up the top half of the border. If this path touches or crosses the line $y = x$ then the figure generated by taking the union of this path with its reflection across $y = x$ will not be a polyomino. Since the first step of our top path must go up, and the last must go to the right, we can ignore these to steps and only consider a path between $(0, 1)$ and $(n - 1, n)$. This path may not touch the line $y = x$. In other words, it may touch, but not cross, the line $y = x + 1$. We can then think of this path as a Dyck path of length $2n - 2$, with northeast steps representing up steps and southeast steps representing steps to the right. The number of Dyck paths of length $n - 1$, and hence the number of paths from $(0, 0)$ to (n, n) that only touch the line $y = x$ at the origin and at (n, n), is C_{n-1}. So the number of parallelogram polyominoes with diagonal symmetry, bounded by an $n \times n$ square is C_{n-1}. □

9.4 With Double-Diagonal Symmetry

Theorem 9.4.1. *There are $\binom{p-1}{\lfloor \frac{p}{2} \rfloor}$ parallelogram polyominoes with double diagonal symmetry of perimeter $4p$.*

Proof. The bounding rectangle for a parallelogram polyomino with double diagonal symmetry must be a square. Since we are considering the set of parallelogram polyominoes with double diagonal symmetry with perimeter $4p$, we know this square has side length p. Call the square $OABC$ as shown below, and define point $P = \overline{AC} \cap \overline{OB}$.

We consider a quadrant of the bounding square created by the two diagonals, for sake of argument the bottom quadrant OPC. A parallelogram

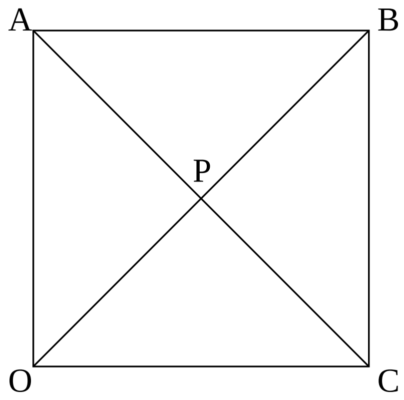

Figure 25

polyomino with double diagonal symmetry is determined uniquely by what is contained in this quadrant through reflection about each diagonal. We require a lattice path from the origin to $\overline{PC}$ which does not touch $\overline{OB}$ except at O and consists of only up and right steps. This is equivalent to the number of paths from $(1,0)$ to $\overline{PC} \setminus \{P\}$.

There are $\sum_{i=0}^{\lfloor \frac{p}{2} \rfloor} \binom{p-1}{i}$ possible paths, counting over the possible number of up steps in the path. We apply the reflection principle to see how many of these cross $\overline{OB}$.

Consider a path from $(1,0)$ to $\overline{PC} \setminus \{P\}$ which crosses $\overline{OB}$ at some point Q. Reflect the path up to point Q over $\overline{OB}$ to get a path from $(0,1)$ to $\overline{PC} \setminus \{P\}$. This reflection is reversible and therefore a bijection between bad paths and paths from $(1,0)$ to $\overline{PC} \setminus \{P\}$.

However, there are $\sum_{i=0}^{\lfloor \frac{p}{2} \rfloor - 1} \binom{p-1}{i}$ paths from $(1,0)$ to $\overline{PC} \setminus \{P\}$, counting based on possible number of up steps, so it follows that the number of "good" paths is

$$\sum_{i=0}^{\lfloor \frac{p}{2} \rfloor} \binom{p-1}{i} - \sum_{i=0}^{\lfloor \frac{p}{2} \rfloor - 1} \binom{p-1}{i} = \binom{p-1}{\lfloor \frac{p}{2} \rfloor},$$

meaning we have $\binom{p-1}{\lfloor \frac{p}{2} \rfloor}$ parallelogram polyominoes with double diagonal symmetry, as desired. □

10 Directed Convex Polyominoes

A *directed convex polyomino* is a polyomino that is both vertically and horizontally convex and satisfies the condition that for every point P in the polyomino, there exists a path consisting only of up and right steps starting at the lower left corner of the polyomino and ending at P. Another definition of directed convex polyominoes is polyominoes that are vertically and horizontally convex with a unique lower left corner.

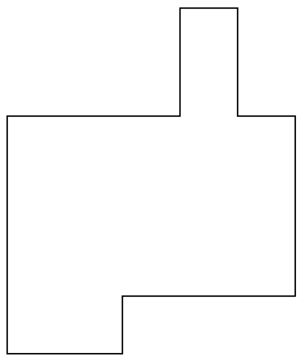

Figure 26: A Directed Convex Polyomino

10.1 By Bounding Rectangle

Theorem 10.1.1. *The number of directed convex polyominoes in an* $m \times n$ *rectangle is* $\binom{m+n-2}{m-1}\binom{m+n-2}{n-1}$

Proof. Suppose without loss of generality that $m \leq n$.
Consider the bounding rectangle in the plane with the lower left corner placed at $(0, 0)$ and the upper right corner at (m, n). Furthermore, consider the line $x + y = m + n - 1$.

We also consider breaking a directed convex polyomino into two paths, each of length $m + n$. One of these paths goes first to $(0, 1)$ (the "up" path), and the other goes first to $(1, 0)$ (the "right" path) or our polyomino would be invalid. The up path must touch the top edge of the rectangle and the right path must touch the right side of the rectangle, both in more than one point. Therefore, we remove the first unit segment where a path touches its

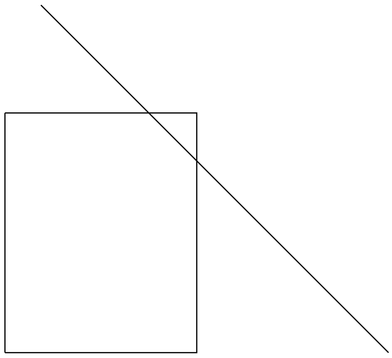

Figure 27

corresponding side, and reflect the rest of the path over that side. Now, each path ends on the line $x + y = m + n - 1$. The top path must end at some point $P = (x, m + n - 1 - x)$ with $x \leq m - 1$, and the right path must end at some point $Q = (m + n - 1 - y, y)$ with $y \leq n - 1$.

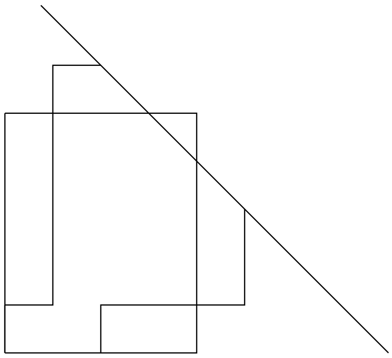

Figure 28

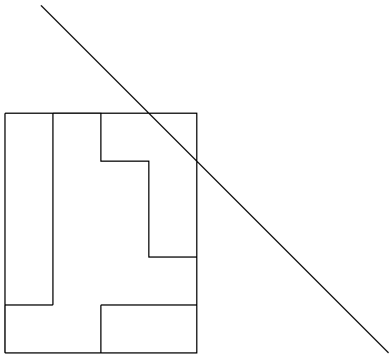

Figure 29

Furthermore, we must have the condition that when P is reflected over the line $y = n$ and translated one unit to the right, it coincides with the image of Q after it is reflected over the line $x = m$ and translated one unit upward.

We can model these reflections and translations algebraically to find

$$P \mapsto (x+1, n-m+1+x) \qquad \text{and} \qquad Q \mapsto (m-n+1+y, y+1),$$

from which we find $y = x+n-m$, so $Q = (2m-x-1, x+n-m)$.
Therefore, Q is determined by P.

We apply the Reflection Principle to determine the number of non-intersecting pairs of paths from $(0,1)$ to some point P and $(1,0)$ to the corresponding point Q.

First we count the total number of possible pairs. The only restriction we have on P is that $x \leq m-1$, so we let x range over integers in $[0, m-1]$. For a fixed x-value, x_0, there are $\binom{m+n-2}{x_0}$ paths from $(0,1)$ to P. Since $Q = (2m-x_0-1, x_0+n-m)$, there are $\binom{m+n-2}{n-m+x_0}$ paths from $(1,0)$ to Q. Therefore, we get

$$\sum_{x=0}^{m-1} \left[\binom{m+n-2}{x} \binom{m+n-2}{n-m+x} \right] \tag{1}$$

total possible paths.

Now we count how many pairs of paths cross. Using the Reflection Principle, this is the number of pairs of paths from $(0,1)$ to Q and $(1,0)$ to P. Note that we cannot freely choose P now since if $P = (0, m+n-1)$ there is no path from $(1,0)$ to P consisting of only up and right moves. Otherwise, we can let x range over integers in $[1, m-1]$. By a similar argument to that above, we find there are $\sum_{x=1}^{m-1} \left[\binom{m+n-2}{x-1} \binom{m+n-2}{n-m+x-1} \right]$ pairs of crossing paths. Letting $j = x-1$ in this equation the summation becomes

$$\sum_{j=0}^{m-2} \left[\binom{m+n-2}{j} \binom{m+n-2}{n-m+j} \right]. \tag{2}$$

Combining (1) and (2), we get that there are $\sum_{x=0}^{m-1} \left[\binom{m+n-2}{x} \binom{m+n-2}{n-m+x} \right] - \sum_{j=0}^{m-2} \left[\binom{m+n-2}{j} \binom{m+n-2}{n-m+j} \right] = \binom{m+n-2}{m-1} \binom{m+n-2}{n-1}$ total pairs of nonintersecting paths, giving us

$$\binom{m+n-2}{m-1} \binom{m+n-2}{n-1}$$

directed convex polyominoes, as desired. □

10.2 With Diagonal Symmetry

Theorem 10.2.1. *The number of diagonally symmetric directed convex polyominoes that can be inscribed in an $n \times n$ square is $\binom{2n-2}{n-1}$.*

Proof. We apply a similar argument to the one for generalized directed convex polyominoes by taking pairs of paths from $(0,0)$ to $x+y=2n-1$. Note that one of these paths uniquely determines the other due to diagonal symmetry, so let us consider the path which begins $(0,0) \mapsto (0,1)$. Note that the path cannot intersect the diagonal of the square until it finishes.

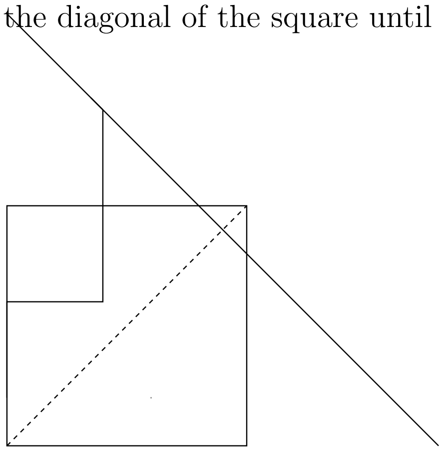

Figure 30

We apply a similar reflection as earlier, removing one unit of the path as the path intersects the top edge of the square and reflecting the remainder of the path over the top edge of the square to get a path from $(0,1)$ to $x+y=2n-1$. Suppose the path ends at a point $P=(x,2n-1-x)$ on this line. Then clearly $0 \leq x \leq n-1$. The total number of such paths is therefore

$$\sum_{x=0}^{n-1} \binom{2n-2}{x}. \tag{3}$$

We again apply the Reflection Principle and note the number of paths which touch the diagonal is equal to the number of paths from $(1,0)$ to some point $P=(x,2n-1-x)$ on $x+y=2n-1$, where $1 \leq x \leq n-1$ (since clearly x cannot be 0). Therefore there are

$$\sum_{x=1}^{n-1} \binom{2n-2}{x-1} = \sum_{j=0}^{n-2} \binom{2n-2}{j} \tag{4}$$

paths which intersect the diagonal.

Putting (3) and (4) together yields a total of

$$\sum_{x=0}^{n-1}\binom{2n-2}{x} - \sum_{x=0}^{n-2}\binom{2n-2}{x} = \binom{2n-2}{n-1}$$

diagonally symmetric convex polyominoes.

□

11 Polyominoes with Diagonal Steps

A *polyomino with diagonal steps* is a modified version of a polyomino in that its boundary is composed of horizontal, vertical, and diagonal steps.

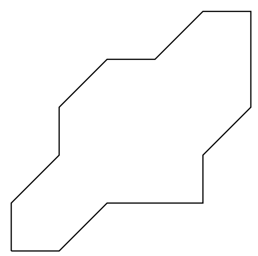

Figure 31: A polyomino with diagonal steps

11.1 With Diagonal Symmetry

Theorem 11.1.1. *The number of diagonally symmetric parallelogram polyominoes with diagonal steps with bounding rectangle* $n\times n$ *is* $\sum_{m=1}^{n} \left(\!\binom{2m-1}{n-m}\!\right) C_{m-1}$

Proof. To generate a diagonally symmetric parallelogram polyomino with diagonal steps with bounding rectangle $n \times n$, we may take $1 \leq m \leq n$ and place a $m \times m$ bounding rectangle in the lower left corner of the $n \times n$ bounding rectangle.

Then we may consider a diagonally symmetry parallelogram polyomino without diagonal steps with $m \times m$ bounding rectangle. By adding diagonal steps to the polyomino, we will have it fill out the $n \times n$ bounding rectangle. Because we are only considering diagonally symmetric polyominoes, we need

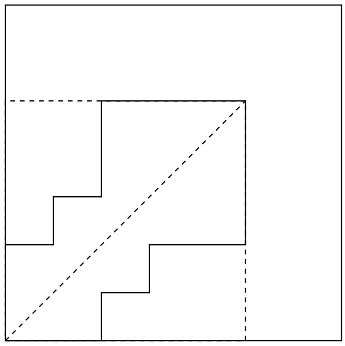

Figure 32

only generate the lower boundary, as we can reflect that path across the line $y = x$ to obtain the whole polyomino. We may insert diagonal steps at any lattice point on the lower boundary except for the bottom left and top right points. Notice that inserting a diagonal step at a lattice point will create a polyomino with $(m + 1) \times (m + 1)$ bounding rectangle. There are $2m - 1$ lattice points on the lower boundary where diagonal steps may be added, and we must insert $n - m$ diagonal steps in the lower boundary to generate a polyomino with an $n \times n$ bounding rectangle.

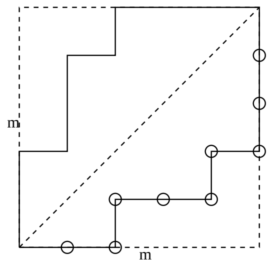

Figure 33: The $2m - 1$ points where a diagonal step may be added

Multiple diagonal steps may be added at any lattice point on the lower boundary, so we count the number of ways to insert the diagonal steps with the multichoose function. So there are $\left(\!\binom{2m-1}{n-m}\!\right)$ ways to insert the diagonal steps. Recall that there are C_{m-1} diagonally symmetric parallelogram polyominoes by Theorem 9.3.1. Thus, there are

$$\sum_{m=1}^{n} \left(\binom{2m-1}{n-m} \right) C_{m-1}$$

diagonally symmetric parallelogram polyominoes with diagonal steps and $n \times n$ bounding rectangle.

□

12 Conclusion

There are many directions still to be explored. Given more time, we would try to count polyominoes with diagonal steps and vertically convex polyominoes more generally, rather than only considering special cases. Counting stacks by area and skyline polyominoes by perimeter are outstanding problems as well. We are still looking for a cleaner, more intuitive bijection to count directed convex polyominoes. Of course, the larger goal of counting general polyominoes by area, perimeter, or bounding rectangle also still remains.

13 Acknowledgments

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