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Art of Problem Solving Introductory Topics
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CHAPTER 1

Three Types of Probability

This article is not so much about particular problems or problem solving tactics as it is about labels. If you think about it, labels are a big key to the way we organize ideas. When we already have the central concepts to problems organized, we are better able to solve them and our solutions are often more efficient. In short, labels help us organize – and organization simplifies problem solving! This article seeks to demonstrate the power of intelligent classification using types of probability as an example.

1.1 Introduction

Each day on his way to work, Steve drives up to the busy four-way intersection in Omaha. When the traffic light signals green, Steve drives through the intersection. When the traffic light is red, he stops and waits for it to turn green. When the traffic light is yellow, Steve considers whether or not he will make it through the intersection in time before making a decision as to whether to stop or go. It doesn't take a whole lot of effort for Steve to make it into work.

Without the traffic light in place, making it through the intersection might be a chore and it might not even be possible. He would always have to slow down, prepared to stop if necessary. He'd need to look around to see if there are cars coming from the other three directions that might cross his path. If there are enough other drivers, the whole process would be chaos!

It's a good thing we have traffic lights to make driving easier.

Now, let's build a probability traffic light! We can classify three main types of probability problems based on the ways in which we can approach them: **counting**, **geometry**, and **algebra**. When we can identify these types as easily as the colors on a traffic light, we can cut to the chase and solve problems.

1.2 Probability as Counting

The first type of probability we will discuss is perhaps the simplest to understand. Let P(event) be the probability of some event occuring. We can often determine P(event) by counting the number of successful outcomes and then dividing by the total number of equally likely outcomes:

Concept: ☞

$$\text{P(event)} = \frac{\text{\# of successful outcomes}}{\text{\# of total outcomes}}$$

Let's take a look at a couple of problems that apply this principle of counting to solve probability problems.

Problem 1.1: Find the probability that when two standard 6-sided dice are rolled, the sum of the numbers on the top faces is 5.

Solution for Problem 1.1: There are $6 \cdot 6 = 36$ possible outcomes when we roll a pair of dice. We can list the outcomes in which the sum of the top faces is 5:

⚀ + ⚃ = 5
⚁ + ⚂ = 5
⚂ + ⚁ = 5
⚃ + ⚀ = 5

We can now reach an answer by dividing the number of successful outcomes by the total number of possible outcomes:

$$\text{P(Sum of 5)} = \frac{\text{\# of successful outcomes}}{\text{\# of total outcomes}} = \frac{4}{36} = \frac{1}{9}.$$

□

Now let's take a look at another example of counting probability that requires a bit more thought:

Problem 1.2: A bag contains 16 marbles, 4 of which are blue and 12 of which are green. Two marbles are randomly pulled from the bag at the same time. What is the probability that both marbles are blue?

Solution for Problem 1.2: We can work this problem in several ways – all of which are based in counting methods.

In our first solution we note that it doesn't matter that both marbles are drawn at once. We can arbitrarily call one of them the *first marble* and the other the *second marble*. There are 16 choices for the first marble, leaving 15 choices for the second marble. The total number of ways in which we can draw two marbles from the bag is thus $16 \cdot 15 = 240$.

Now we count the number of ways in which we can draw two blue marbles from the bag in two draws. There are 4 choices for the first blue marble, leaving 3 for the second blue marble, for a total of $4 \cdot 3 = 12$ ways in which we can draw two blue marbles from the bag.

We now calculate the probability:

$$P(2 \text{ blue marbles}) = \frac{\#\text{ of ways to draw 2 blue marbles}}{\#\text{ of ways to draw 2 marbles}} = \frac{12}{240} = \frac{1}{20}.$$

Students familiar with combinations might solve this problem in a related manner. If we don't consider the order in which we select the marbles, we can note that there are $\binom{16}{2}$ ways in which we can select two marbles and $\binom{4}{2}$ ways in which we can select two blue marbles. Now our calculation looks like this:

$$\begin{aligned} P(2 \text{ blue marbles}) &= \frac{\#\text{ of ways to draw 2 blue marbles}}{\#\text{ of ways to draw 2 marbles}} \\ &= \frac{\binom{4}{2}}{\binom{16}{2}} = \frac{\frac{4\cdot 3}{2}}{\frac{16\cdot 15}{2}} = \frac{6}{120} = \frac{1}{20} \end{aligned}$$

We could even solve this problem by calculating the probability one draw at a time and multiplying. The probability that the first marble is blue is $\frac{4}{16} = \frac{1}{4}$. If the first marble was blue, then 3 of the remaining 15 marbles are blue, so the probability that the second marble will also be blue is $\frac{3}{15} = \frac{1}{5}$. The probability that both are blue is the product of the probabilities of each draw:

$$\begin{aligned} P(2 \text{ blue marbles}) &= P(\text{first marble is blue}) \cdot P(\text{second marble is blue}) \\ &= \frac{1}{4} \cdot \frac{1}{5} = \frac{1}{20} \end{aligned}$$

□

Counting probability problems are fairly common and they are the easiest to recognize. When we can count successful and total numbers of equally likely outcomes, the probability is the ratio between the two.

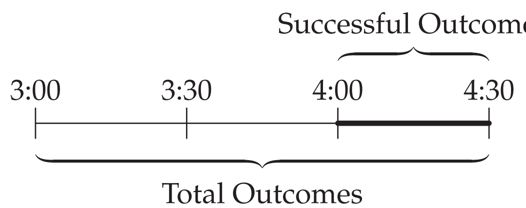
1.3 Probability as Geometry

Sometimes it is impossible to count the total number of events because there are infinitely many possibilities. Some problems involve evaluating points on a line segment or in space, or particular values of a continuous variable such as time. The following is a simple example:

Problem 1.3: Lawrence parked his car in a parking lot at a randomly chosen time between 2:30 PM and 4:00 PM. Exactly half an hour later he drove his car out of the parking lot. What is the probability that he left the parking lot after 4:00 PM?

Solution for Problem 1.3: We can divide hours into minutes, minutes into seconds, and seconds into smaller and smaller fractions. There are an infinite number of times during which Lawrence could have driven away. We cannot count times to solve this problem!

Time is continuous, so we'll have to think differently about this problem. Let's take a look at a line segment that represents the possible times Lawrence could drive away:



We can determine the probability that Lawrence drove away after 4:00 PM by comparing the length of the segment of successful outcomes to that of the total outcomes:

$$\text{P(success)} = \frac{\text{length of successful segment}}{\text{length of total segment}} = \frac{30 \text{ (minutes)}}{90 \text{ (minutes)}} = \frac{1}{3}.$$

□

Now that we've seen one example of geometric probability, we can generalize the central idea:

Concept: ☞ When a probability calculation involves one or more continuous variables, we can determine the probability by comparing the sizes of geometric regions:

$$\text{P(event)} = \frac{\text{size of successful region}}{\text{size of total region}}.$$

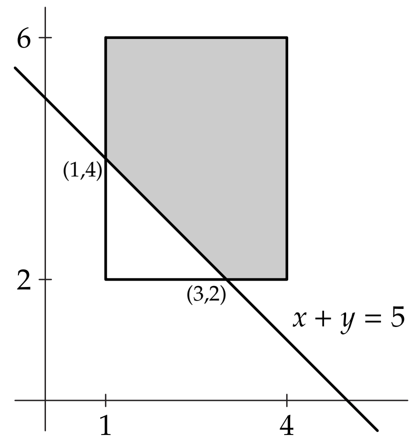
Sometimes the sizes we compare are lengths, sometimes they are areas, and sometimes they are volumes.

Let's take a look at a problem involving multiple continuous variables:

Problem 1.4: If $1 \leq x \leq 4$ and $2 \leq y \leq 6$, find the probability that $x + y \geq 5$.

Solution for Problem 1.4: The problem is a little more complex because we have two distinct continuous variables. However, the concept is the same: we need to compare the successful region to the total region in order to determine the probability.

Let's examine the graph of the total region in which the successful region is shaded:



The regions we are comparing are areas. We determine the probability by comparing these areas:

$$\frac{\text{area of successful region}}{\text{area of total region}} = \frac{10}{12} = \frac{5}{6}.$$

Some adept problem solvers might notice that it's easier to compute the area of the unsuccessful region. We can calculate that probability and subtract it from 1:

$$\text{P(successful outcome)} = 1 - \text{P(unsuccessful outcome)} = 1 - \frac{2}{12} = \frac{5}{6}.$$

We call this technique **complementary probability** and it comes in handy quite often. □

1.4 Probability as Algebra

While it is rarely (if ever) taught in regular textbooks, we can solve some probability problems using algebraic techniques. The idea behind the next couple of problems is to view probabilities as variables. We then apply algebraic methods to find the values of those variables.

Let's take a look at a fairly simple example:

Problem 1.5: If the probability that it rains next Tuesday in Seattle is twice the probability that it doesn't, what is the probability that it rains next Tuesday in Seattle?

Solution for Problem 1.5: We can neither count the rain nor represent it as a continuous variable. However, we can name the probability that it rains next Tuesday. That is, we can assign its value a variable.

Let x be the probability that it rains next Tuesday. We can now translate this word problem into a math problem in terms of x. Since it either will rain or won't rain next Tuesday in Seattle, the probability that it won't must be $1 - x$. We are told that

$$x = 2(1 - x).$$

Solving this equation we find that $x = \frac{2}{3}$, which is our answer. □

Concept: ☞ If you can identify enough relationships between probabilities, you might be able to solve the problem algebraically.

Let's take a look at a problem that requires a little more insight into algebraic relationships:

Problem 1.6: Johnny and Michael play a game in which they take turns rolling a pair of fair dice until one of them rolls "snake eyes" (both dice show 1's). That person is the winner. If Johnny goes first, what is the probability that Johnny wins the game?

Solution for Problem 1.6: Let's go ahead and name the values of the probabilities that Johnny wins and that Michael wins. Let j be the probability that Johnny wins and m be the probability that Michael wins.

Since the probability that one of the two wins is 1 (make sure you see why the probability that the game is never won is 0), we have the equation $j + m = 1$. Unfortunately, this alone is not enough to solve the problem.

It often helps to explore some simple cases of a problem when you don't immediately see a path toward the solution. By examining the probabilities that Johnny and Michael each win during their first toss, second toss, etc., we notice that each of their probabilities is the sum of a geometric series:

$$\begin{array}{ccccccccc} & & \text{winner} & & \text{winner} & & \text{winner} & & \dots \\ & & \underline{\text{on toss 1}} & & \underline{\text{on toss 2}} & & \underline{\text{on toss 3}} & & \\ j & = & \frac{1}{36} & + & \frac{1}{36}\left(\frac{35}{36}\right)^2 & + & \frac{1}{36}\left(\frac{35}{36}\right)^4 & + & \cdots \\ m & = & \frac{1}{36}\left(\frac{35}{36}\right) & + & \frac{1}{36}\left(\frac{35}{36}\right)^3 & + & \frac{1}{36}\left(\frac{35}{36}\right)^5 & + & \cdots \end{array}$$

While we could now find the value of j by summing a geometric series, a new relationship has presented itself! Notice that the ratio between the probability that Johnny wins and the probability that Michael wins is constant for every toss. This means that m is $\frac{35}{36} \cdot j$. We now have a system of linear equations in two variables:

$$\begin{aligned} j + m &= 1 \\ m &= \frac{35}{36} \cdot j \end{aligned}$$

We can solve this system by ordinary means to find that $j = \frac{36}{71}$. □

1.5 Summary

Probability can get much harder than the problems we explored. However, our categorization of the areas of math that we can use to tackle these problems helps demystify probability at any level. Once we recognize these problems for what they are – counting, geometry, and algebra – we can be more confident and straight-forward in choosing the ways in which we approach probability problems.

Now that you see the power of intelligent classification, you can try to apply it to other areas of mathematics and other problem solving subjects!

1.6 Some Probability Practice Problems

Throughout the following probability problems, try first to determine the nature of the problem – whether it is a problem that can be solved using counting techniques, geometric techniques, algebraic techniques, or possibly a combination of more than one. Then try to solve each problem.

Exercises for Section 1.6

1.6.1 Olga and Andrew play chess each day. The probability that Olga beats Andrew on any given day is twice the probability that Andrew beats Olga. The probability that Andrew beats Olga is three times the probability that the game ends in a draw (tie). What is the probability that any particular game between Olga and Andrew ends in a draw?

1.6.2 A fair coin is flipped three times. What is the probability that at least one lands heads and at least one lands tails?

1.6.3 Of the 12 marbles in a bag, exactly 3 are blue. If Sandor reaches into the bag and pulls out two marbles, what is the probability that exactly one of them is blue?

1.6.4 Two real numbers x and y are randomly selected on a number line between 0 to 12. Find the probability that their sum is greater than 15.

1.6.5 The surface of an $8 \times 10 \times 12$ block is painted green after which the block is cut up into 960 smaller $1 \times 1 \times 1$ cubes. If one of the smaller cubes is selected at random, what is the probability that it has green paint on at least one of its faces?

1.6.6 Ted and Erin play a game in which they take turns flipping a fair coin. The first player to flip heads wins. If Erin goes first, what is the probability she wins?

1.6.7 Bender orders two robot painters to paint a 1000 meter long circular fence. One robot painter is ordered to paint a randomly selected 720 meter stretch of the fence while the other is ordered to paint a randomly selected 750 meter stretch of the fence. Find the probability that when the robot painters are done, the entire fence has been painted.

1.6.8 A point (x, y) in the coordinate plane satisfies the inequalities $-6 < x < 6$ and $-6 < y < 6$, where x and y are real numbers. Find the probability that the point (x, y) is less than 4 units from the origin.

1.6.9★ A lattice point (x, y) is selected at random from the interior of the circle described by the equation $x^2 + y^2 - 2x + 4y = 22$. Find the probability that the point is within 1.5 units from the origin.

1.6.10★ An 'unfair' coin has a 2/3 probability of turning up heads. If this coin is tossed 50 times, what is the probability that the total number of heads is even? *(Source: AHSME)*

1.6.11★ Two mathematicians take a morning coffee break each day. They arrive at the cafeteria independently, at random times between 9 AM and 10 AM, and stay for exactly m minutes. Find the value of m if the probability that either one arrives while the other is in the cafeteria is 40%. *(Source: AIME)*

1.6.12★ A fair coin is to be tossed ten times. Find the probability that heads never occur on consecutive tosses. *(Source: AIME)*

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