Reminder: The AM-GM Inequality for two variables states that if a and b are nonnegative, then

$$\frac{a+b}{2} \geq \sqrt{ab}.$$

- Applying AM-GM to 1 and x, we have $\frac{1+x}{2} \geq \sqrt{(1)(x)}$. Multiplying both sides by 2 gives $1 + x \geq 2\sqrt{x}$.

- We work backwards to find our solution. We multiply both sides of the desired inequality by 2 and expand the right side to give us $2x^2 + 2y^2 \geq x^2 + 2xy + y^2$. Rearranging gives $x^2 - 2xy + y^2 \geq 0$, or $(x - y)^2 \geq 0$. Now we have a path to our solution.

  By the Trivial Inequality, we have $(x-y)^2 \geq 0$. Expanding the left side and rearranging gives $x^2 + y^2 \geq 2xy$. Adding $x^2 + y^2$ to both sides gives $2x^2 + 2y^2 \geq x^2 + 2xy + y^2$. The right side is the square of $x + y$, so we have $2x^2 + 2y^2 \geq (x + y)^2$. Dividing by 2 gives the desired $x^2 + y^2 \geq \frac{(x+y)^2}{2}$.

- We work backwards to find our solution. Multiplying both sides by $xy(x+y)$ (which doesn't change the direction of the inequality because x and y are positive), gives us

  $$y(x + y) + x(x + y) \geq 4xy.$$

  Rearranging this gives us $x^2 + y^2 - 2xy \geq 0$, which we know how to handle with the Trivial Inequality. Now we are ready to write our solution.

  By the Trivial Inequality, we have $(x - y)^2 \geq 0$. Expanding the left side, then adding $4xy$ to both sides, gives $x^2+2xy+y^2 \geq 4xy$. We rewrite the left side to give $x^2+xy+y^2+xy \geq 4xy$, and a little factoring gives us $x(x + y) + y(x + y) \geq 4xy$. Dividing both sides by $xy(x + y)$ (which doesn't change the direction of the inequality because x and y are positive), gives us the desired $\frac{1}{x} + \frac{1}{y} \geq \frac{4}{x+y}$.

- The sums on the larger side make us think of AM-GM. We have

  $$\frac{x+y}{2} \geq \sqrt{xy},$$
  $$\frac{y+z}{2} \geq \sqrt{yz},$$
  $$\frac{z+x}{2} \geq \sqrt{zx}.$$

  Multiplying these three gives us

  $$\frac{(x+y)(y+z)(z+x)}{8} \geq \sqrt{x^2y^2z^2}.$$

  Because $\sqrt{x^2y^2z^2} = xyz$, multiplying our inequality by 8 gives us the desired inequality $(x + y)(y + z)(z + x) \geq 8xyz$.

- We put all the terms on one side, which gives us

$$x^2 + y^2 - xy - x - y + 1 \geq 0.$$

The $x^2 + y^2 - xy$ gets us thinking about $(x-y)^2$, which equals $x^2 + y^2 - 2xy$. We need only $-xy$, not $-2xy$, so we consider

$$\frac{(x-y)^2}{2} = \frac{x^2 + y^2 - 2xy}{2} = \frac{x^2 + y^2}{2} - xy.$$

That takes care of the xy term, but now we need another $\frac{x^2}{2}$ and $\frac{y^2}{2}$. We can get an x^2 term and an x term from $(x-1)^2$, which will also give us a 1. We only want $x^2/2$, so we consider $(x-1)^2/2$ (and $(y-1)^2/2$):

$$\frac{(x-1)^2}{2} = \frac{x^2 - 2x + 1}{2} = \frac{x^2}{2} + \frac{1}{2} - x,$$
$$\frac{(y-1)^2}{2} = \frac{y^2 - 2y + 1}{2} = \frac{y^2}{2} + \frac{1}{2} - y.$$

Now we have all our terms and we can use the Trivial Inequality to finish. The Trivial Inequality gives us

$$\frac{(x-y)^2}{2} + \frac{(x-1)^2}{2} + \frac{(y-1)^2}{2} \geq 0,$$

and expanding the terms on the left side (and collecting like terms) gives $x^2 + y^2 - xy - x - y + 1 \geq 0$. A little rearranging then gives the desired $x^2 + y^2 \geq xy + x + y - 1$.

- This looks a little like AM-GM, but with 4 variables instead of 2. So, let's start by writing AM-GM twice:

$$\frac{w+x}{2} \geq \sqrt{wx},$$
$$\frac{y+z}{2} \geq \sqrt{yz}.$$

If we add these, we'll have $w + x + y + z$ on the left:

$$\frac{w+x+y+z}{2} \geq \sqrt{wx} + \sqrt{yz}.$$

We can get the product $wxyz$ by applying AM-GM to $\sqrt{wx}$ and $\sqrt{yz}$:

$$\frac{\sqrt{wx} + \sqrt{yz}}{2} \geq \sqrt{\sqrt{wx}\sqrt{yz}}.$$

Simplifying the right side gives us $\sqrt[4]{wxyz}$ on the right, then multiplying by 2 gives us

$$\sqrt{wx} + \sqrt{yz} \geq 2\sqrt[4]{wxyz}.$$

Combining this with our earlier inequality for $w + x + y + z$ gives us

$$\frac{w+x+y+z}{2} \geq \sqrt{wx} + \sqrt{yz} \geq 2\sqrt[4]{wxyz},$$

so we have

$$\frac{w+x+y+z}{2} \geq 2\sqrt[4]{wxyz}.$$

Dividing this by 2 gives us the desired

$$\frac{w+x+y+z}{4} \geq \sqrt[4]{wxyz}.$$

For a much harder challenge, try proving $(x + y + z)/3 \geq \sqrt[3]{xyz}$ for all nonnegative x, y, and z.